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Improved Bounds on the Effective Elastic Moduli of Random Arrays of Cylinders

Improved rigorous bounds on the effective elastic moduli of a transversely isotropic fiber-reinforced material composed of aligned, infinitely long, equisized, circular cylinders distributed throughout a matrix are evaluated for cylinder volume fractions up to 70 percent. The bounds are generally shown to provide significant improvement over the Hill-Hashin bounds which incorporate only volume-fraction information. For cases in which the cylinders are stiffer than the matrix, the improved lower bounds provide relatively accurate estimates of the elastic moduli, even when the upper bound diverges from it (i.e., when the cylinders are substantially stiffer than the matrix). This last statement is supported by accurate, recently obtained Monte Carlo computer-simulation data of the true effective axial shear modulus.

1 Introduction

In earlier works (Torquato and Lado, 1988a,b) we computed bounds on the effective axial shear modulus μ_e (or, equivalently, transverse conductivity) and the effective transverse bulk modulus k_e of random arrays of infinitely long, oriented cylindrical fibers in a matrix that improved upon the well-known second-order bounds of Hill (1964) and Hashin (1965, 1970). These third and fourth-order bounds depend upon the key microstructural parameter ζ_2—a multidimensional integral, which was evaluated for the aforementioned model by us in the superposition approximation. Recent computer simulations (described below) indicate that ζ_2 in the superposition approximation is not accurate at high-fiber volume fractions. Moreover, corresponding improved bounds on the effective transverse shear modulus G_e, that depend upon a different microstructural parameter η_2, of random arrays of cylinders have heretofore not been computed.

Rigorous upper and lower bounds on the effective properties of composite materials are useful because: (i) they enable one to test the merits of theories; (ii) as successively more microstructural information is incorporated, the bounds become progressively narrower; (iii) one of the bounds can typically provide a relatively accurate estimate of the property even when the reciprocal bound diverges from it (Torquato, 1985, 1987), a point that has yet to be fully appreciated.

The purpose of this paper is to determine accurate analytical expressions for both ζ_2 and η_2 and thus improved bounds on μ_e, k_e, and G_e for the practically useful model of impenetrable, parallel, infinitely long, equisized, circular cylinders (or circular disks in two dimensions) distributed randomly throughout a matrix. Accurate computer simulations of the true effective axial shear modulus μ_e recently obtained by Kim and Torquato (1990) shall be used to access the accuracy of the improved bounds on μ_e. Note that transversely isotropic, fiber-reinforced materials are characterized by five effective elastic constants but Hill (1964) showed that for such two-component materials it is only necessary to determine three of the constants since the other two can then be easily calculated.

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2 Improved Bounds on μ_e, k_e, and G_e

Given only the phase volume fractions, ϕ_1 and ϕ_2, bulk moduli, K_1 and K_2, and shear moduli, G_1 and G_2, of a two-phase transversely isotropic fiber-reinforced material, Hill (1964) and Hashin (1965, 1970) have derived the best possible bounds on μ_e, k_e, and G_e. Silnutzer (1972) derived improved bounds on μ_e and k_e that additionally depend upon two integrals over certain three-point correlation functions. Milton (1982) showed that both integrals can be expressed in terms of a single integral ζ_2 defined as follows:

$$\zeta_2=\frac{4}{\pi\phi_1\phi_2}\int_0^\infty\frac{dr}{r}\int_0^\infty\frac{ds}{s}\int_0^\pi d\theta\left[S_3(r,s,t)-\frac{S_2(r)S_2(s)}{\phi_2}\right]\cos2\theta. \tag{1}$$

The quantities $S_2(r)$ and $S_3(r,s,t)$ are, respectively, the probabilities of finding in phase 2 the end points of a line segment of length r and the vertices of a triangle with sides of length r, s and t; θ is the included angle opposite the side of length t. The Silnutzer bounds are referred to as third-order bounds since they are exact up to third order in the difference in the phase properties.

Silnutzer also derived third-order bounds on the effective transverse shear modulus G_e which Milton showed could be expressed in terms of ζ_2 another microstructural parameter η_2 where

$$\eta_2=\frac{16}{\pi\phi_1\phi_2}\int_0^\infty\frac{dr}{r}\int_0^\infty\frac{ds}{s}\int_0^\pi d\theta\left[S_3(r,s,t)-\frac{S_2(r)S_2(s)}{\phi_2}\right]\cos 4\theta. \quad (2)$$

The simplified forms of the Silnutzer third-order bounds (Milton, 1982) for the axial shear modulus, transverse bulk modulus, and transverse shear modulus are, respectively, given by

$$\mu_L^{(3)}\le\mu_e\le\mu_U^{(3)}, \quad (3)$$

where

$$\mu_U^{(3)}=\left[\langle G\rangle-\frac{\phi_1\phi_2(G_2-G_1)^2}{\langle\tilde{G}\rangle+\langle G\rangle_\zeta}\right], \quad (4)$$

$$\mu_L^{(3)}=\left[\langle 1/G\rangle-\frac{\phi_1\phi_2(1/G_2-1/G_1)^2}{\langle 1/\tilde{G}\rangle+\langle 1/G\rangle_\zeta}\right]^{-1}; \quad (5)$$

$$k_L^{(3)}\le k_e\le k_U^{(3)}, \quad (6)$$

where

$$k_U^{(3)}=\left[\langle k\rangle-\frac{\phi_1\phi_2(k_2-k_1)^2}{\langle\tilde{k}\rangle+\langle G\rangle_\zeta}\right], \quad (7)$$

$$k_L^{(3)}=\left[\langle 1/k\rangle-\frac{\phi_1\phi_2(1/k_2-1/k_1)^2}{\langle 1/\tilde{k}\rangle+\langle 1/G\rangle_\zeta}\right]^{-1}; \quad (8)$$

and

$$G_L^{(3)}\le G_e\le G_U^{(3)}, \quad (9)$$

where

$$G_U^{(3)}=\left[\langle G\rangle-\frac{\phi_1\phi_2(G_2-G_1)^2}{\langle\tilde{G}\rangle+\Theta}\right], \quad (10)$$

$$G_L^{(3)}=\left[\langle 1/G\rangle-\frac{\phi_1\phi_2(1/G_2-1/G_1)^2}{\langle 1/\tilde{G}\rangle+\Xi}\right]^{-1}, \quad (11)$$

$$\Theta=[2\langle k\rangle_\zeta\langle G\rangle^2+\langle k\rangle^2\langle G\rangle_\eta]/\langle k+2G\rangle^2, \quad (12)$$

$$\Xi=2\langle 1/k\rangle_\zeta+\langle 1/G\rangle_\eta. \quad (13)$$

Here we define $\langle b\rangle=b_1\phi_1+b_2\phi_2$, $\langle\tilde{b}\rangle=b_1\phi_2+b_2\phi_1$, $\langle b\rangle_\zeta=b_1\zeta_1+b_2\zeta_2$, and $\langle b\rangle_\eta=b_1\eta_1+b_2\eta_2$, where $\zeta_1=1-\zeta_2$ and $\eta_1=1-\eta_2$. The quantity k_i is the transverse bulk modulus of phase i for transverse compression without axial extension and may be expressed in terms of the isotropic phase moduli as $k_i=K_i+G_i/3$ ($i=1,2$). Since the microstructural parameters ζ_i and η_i lie in the closed interval [0,1], the above third-order bounds always improve upon the second-order bounds of Hill and Hashin. Note that the problem of determining the effective axial shear modulus μ_e is mathematically equivalent to determining the effective transverse thermal or electrical conductivity (Hashin, 1970).

Fourth-order bounds on the effective transverse conductivity or axial shear modulus μ_e of transversely isotropic fiber-reinforced materials that depend upon the phase properties, ϕ_2, ζ_2, and an integral over the four-point probability function S_4 have been derived by Milton (1981). Employing a phase-interchange theorem, Milton demonstrated that the integral involving S_4 can be expressed in terms of ϕ_2 and ζ_2 only. The fourth-order bounds, for the case $G_2\ge G_1$, are given by

$$\mu_L^{(4)}\le\mu_e\le\mu_U^{(4)}, \quad (14)$$

where

$$\mu_U^{(4)}=G_2\left[\frac{(G_1+G_2)(G_1+\langle G\rangle)-\phi_2\zeta_1(G_2-G_1)^2}{(G_1+G_2)(G_2+\langle\tilde{G}\rangle)-\phi_2\zeta_1(G_2-G_1)^2}\right] \quad (15)$$

$$\mu_L^{(4)}=G_1\left[\frac{(G_1+G_2)(G_2+\langle G\rangle)-\phi_1\zeta_2(G_2-G_1)^2}{(G_1+G_2)(G_1+\langle\tilde{G}\rangle)-\phi_1\zeta_2(G_2-G_1)^2}\right]. \quad (16)$$

3 Determination of the Parameters ζ_2 and η_2 for Random Arrays of Impenetrable Cylinders

Evaluation of the integrals (1) and (2) requires knowledge of the three-point probability function S_3 for infinitely long, oriented, equisized, impenetrable cylinders (disks in two dimensions). Torquato and Stell (1982) have given an exact infinite series representation of the general n-point function S_n for a two-phase system of arbitrary dimensionality consisting of inclusions (phase 2) distributed throughout a matrix (phase 1). In the special case of a statistically isotropic distribution of identical, impenetrable disks (infinitely long, parallel cylinders) of radius a at an area (volume) fraction ϕ_2, the infinite series for S_n terminates with the nth term (Torquato and Stell, 1985); in the case $n=3$, it is given by

$$S_3(r_{12},r_{13},r_{23})=S_3^{(1)}(r_{12},r_{13},r_{23})+S_3^{(2)}(r_{12},r_{13},r_{23})+S_3^{(3)}(r_{12},r_{13},r_{23}), \quad (17)$$

where

$$S_3^{(1)}=\rho\int m(r_{14})m(r_{24})m(r_{34})d\mathbf{r}_4, \quad (18)$$

$$S_3^{(2)}=\rho^2\int\int m(r_{14})m(r_{24})m(r_{35})g_2(r_{45})d\mathbf{r}_4d\mathbf{r}_5+\rho^2\int\int m(r_{14})m(r_{25})m(r_{34})g_2(r_{45})d\mathbf{r}_4d\mathbf{r}_5+\rho^2\int\int m(r_{15})m(r_{24})m(r_{34})g_2(r_{45})d\mathbf{r}_4d\mathbf{r}_5, \quad (19)$$

$$S_3^{(3)}=\rho^3\int\int\int m(r_{14})m(r_{25})m(r_{36})g_3(r_{45},r_{46},r_{56})d\mathbf{r}_4d\mathbf{r}_5d\mathbf{r}_6, \quad (20)$$

with

$$m(r)=\begin{cases}1, & r<a\\ 0 & r>a.\end{cases} \quad (21)$$

Here, ρ is the number density of disks (cylinders) and therefore the disk area fraction (or cylinder volume fraction) is $\phi_2=\rho\pi a^2$. Moreover, g_2 is the pair (radial) distribution function and g_3 the triplet distribution function for the particles, while $r_{ij}\equiv|\mathbf{r}_j-\mathbf{r}_i|$. The domain of integration in each of the above integrals in (18)–(20) is the infinite area of the macroscopic sample. Note that S_2 can be obtained from the expression for S_3, Eq. (17), by letting two of the three points coincide.

The combination of (17)–(21) and the key integrals for ζ_2 and η_2 (Eqs. (1) and (2), respectively) shows that one must perform fivefold, sevenfold, and ninefold integrations. For the case of ζ_2, these "cluster" integrals have been greatly simplified by Torquato and Lado (1988a) by expanding angle-dependent terms in the integrands in circular harmonics (i.e., Chebyshev polynomials) and using the orthogonality properties of this basis set. No such simplification of the cluster integrals involved in η_2 has thus far been made. However, before simplifying and evaluating η_2, it is instructive to first describe existing results for ζ_2.

After simplification, Torquato and Lado (1988a) found that for impenetrable disks of unit diameter one has exactly

$$\zeta_2=\frac{2}{\pi\phi_1}[b_2\phi_2+b_3\phi_2^2], \quad (22)$$

where

$$b_2=\frac{\pi}{4}\int_1^\infty dr\frac{rg_2(r)}{(r^2-1/4)^2} \quad (23)$$

and

$$b_3=\sum_{n=2}^\infty\frac{(n-1)}{4^{n-3}}\int_1^\infty\frac{dr}{r^{n-1}}\int_1^r\frac{ds}{s^{n-1}}\int_0^\pi d\theta[g_3(r,s,t)-g_2(r)g_2(s)]T_n(\cos\theta). \quad (24)$$

In Eq. (24), $T_n(\cos\theta)=\cos(n\theta)$ is the Chehyshev polynomial of the first kind. Note that the original fivefold, sevenfold, and ninefold cluster integrals have been reduced to manageable one and threefold integrals. Observe also that b_2 and b_3 are complicated functions of ϕ_2 as a result of the appearance of the density-dependent quantities g_2 and g_3. Although g_2 is known accurately in the Percus-Yevick approximation (Lado, 1968), g_3 is more problematical and hence Torquato and Lado (1988a) resorted to the commonly used superposition approximation (Hansen and McDonald, 1986)

$$g_3(r_{12},r_{13},r_{23})\approx g_2(r_{12})g_2(r_{13})g_2(r_{23}) \tag{25}$$

to evaluate the integrals of (24). Recent computer simulation studies (Miller and Torquato, 1990) have shown that use of the superposition approximation gives accurate results provided that ϕ_2 is not large. A striking finding is that the exact expansion of ζ_2 through second order in ϕ_2 yields excellent agreement with the simulation results for a very wide range of volume fractions, i.e., up to the disorder-order phase transition, which for disks occurs at $\phi_2\approx 0.71$ (Wood, 1968). This indicates that cubic and higher-order terms make negligible contributions to ζ_2 (and to the closely related parameter η_2), even at high densities. For disks this exact expansion

$$\zeta_2=\frac{\phi_2}{3}-0.05707\phi_2^2, \tag{26}$$

was also given by Torquato and Lado (1988a), but was not used by them to compute bounds. Therefore, relation (26) is now employed in this study for the range $0\leq\phi_2\leq 0.7$. Note that $\phi_2=0.7$ corresponds to approximately 87 percent of the maximum random close-packing volume fraction, $\phi_2^c\simeq 0.81$ (Stillinger, DiMarzio, and Kornegay, 1964).

As already noted, η_2 has yet to be computed for random impenetrable disks. The first step is to simplify the cluster integrals of η_2 which result after substituting (17)–(20) into (2). Following the expansion procedure of Torquato and Lado (1988a), the integral (2) in conjunction with the integrals (18)–(20) can be simplified (after considerable algebra) to yield the exact result that

$$\eta_2=\frac{8}{\pi\phi_1}\,[c_2\phi_2+c_3\phi_2^2], \tag{27}$$

where

$$c_2=\int_1^\infty drrg_2(r)\,W(r), \tag{28}$$

$$W(r)=\frac{\pi}{4(r^2-1/4)^2}\left[1-\frac{3}{4(r^2-1/4)}-\frac{9}{64(r^2-1/4)^2}+\frac{9}{64(r^2-1/4)^3}+\frac{15}{512(r^2-1/4)^4}\right], \tag{29}$$

$$c_3=\sum_{n=2}^{\infty}\int_1^\infty drr\int_1^r dss\int_0^\pi d\theta[g_3(r,s,t)-g_2(r)g_2(s)]Q_n(r,s,\cos\theta) \tag{30}$$

and

$$Q_n(r,s,\cos\theta)=\frac{1}{4^{n-4}r^{n-2}s^{n-2}}\left\{\frac{(n-1)}{16r^2s^2}+(n-3)\times\left[(n-2)-\frac{n(n-1)}{8r^2}\right]\left[(n-2)-\frac{n(n-1)}{8s^2}\right]\right\}T_n(\cos\theta). \tag{31}$$

By virtue of the appearance of g_2 and g_3, the coefficients c_2 and c_3 of (27) are dependent upon the particle fraction ϕ_2.

We now evaluate the parameter η_2 exactly through second order in ϕ_2. The radial distribution function for impenetrable disks of unit diameter is known exactly through first order in ϕ_2 (Hansen and McDonald, 1986),

$$g_2(r)=H(r-1)[1+A_2(r)\phi_2], \tag{32}$$

where

$$A_2(r)=\frac{4}{\pi}\left[\pi-2\sin^{-1}\left(\frac{r}{2}\right)-r\left(1-\frac{r^2}{4}\right)^{1/2}\right]H(2-r), \tag{33}$$

$$H(x)=\begin{cases}0, & x<0\\ 1, & x>0.\end{cases} \tag{34}$$

Substitution of (32) into (28) yields

$$c_2=\frac{7}{81}\,\pi+0.1387\phi_2, \tag{35}$$

which is exact to the order (and number of significant figures) indicated. The coefficient 0.1387 was determined using a one-dimensional trapezoidal rule. The zero-density limit of g_3 is given by the expression

$$g_3(r_{12},r_{13},r_{23})=H(r_{12}-1)H(r_{13}-1)H(r_{23}-1). \tag{36}$$

Substitution of (36) into (30) and use of the Gaussian-Chebyshev quadrature technique, employed by Torquato and Lado (1988a) to compute ζ_2, gives

$$c_3=-0.3934, \tag{37}$$

which is exact to zeroth order in ϕ_2 (to the number of significant figures indicated). Finally, combination of (27), (35), and (37) yields, to second order in ϕ_2, that

$$\eta_2=\frac{56}{81}\,\phi_2+0.0428\phi_2^2. \tag{38}$$

In accordance with the discussion above, we shall employ result (38) in third-order bounds for the effective transverse shear modulus G_e, Eq. (9), for the range $0\leq\phi_2\leq 0.7$. Note that to second order in ϕ_2, $\eta_2>\zeta_2$ for our model.

In Figs. 1 and 2 we compare our results for the three-point parameters ζ_2 and η_2 for our random model to the few calculations of these quantities for other random microstructures. These include the symmetric-cell material (SCM) with cylindrical cells (Beran and Silnutzer, 1971) for which $\zeta_2=\eta_2=\phi_2$ (Milton, 1981) and equisized, fully penetrable cylinders (FPC), i.e., randomly centered cylinders (Torquato and Beasley, 1986a,b; Joslin and Stell, 1986a). Joslin and Stell (1986b) showed that the parameters for the FPC model were insensitive to polydispersivity in size. (Symmetric-cell materials (Miller, 1969) are constructed by partitioning the space into cells of possibly varying shapes and sizes, with cells randomly designated as phase 1 or phase 2 with probabilities ϕ_1 and ϕ_2, respectively.) Note that the linear terms in the respective volume-fractions expansions of the parameters about $\phi_2=0$ either exactly or approximately represent ζ_2 and η_2 over the allowable volume fraction range (Torquato and Lado, 1988a). Torquato and Lado (1986) have already given the reasons why the values of ζ_2 and η_2 for distributions of fully penetrable particles always lie above the corresponding results for impenetrable particles, which unlike the former do not form clusters until the maximum allowable random close-packing fraction is reached.

4 Calculation of Improved Bounds on μ_e, k_e, and G_e for Random Arrays of Impenetrable Cylinders

Here we shall utilize the results for the three-point parameters, relations (26) and (38), to compute the improved bounds of Section 2 for equilibrium distributions of infinitely long, oriented, impenetrable cylinders. We will consider cases in which the cylindrical fibers are stiffer than the matrix for the fiber volume range $0\leq\phi_2\leq 0.7$. It is now well known (Torquato, 1985; 1987; 1988a) that in such cases that the lower bounds (rather than the upper bounds) will provide a relatively

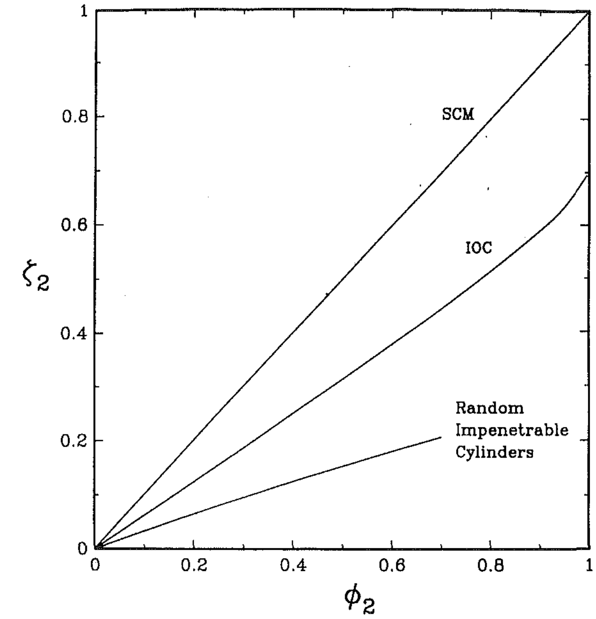

Fig. 1 Three-point parameter ζ_2, defined by Eq. (1), for random arrays of cylinders, including the symmetric-cell material (SCM) with cylindrical cells (Beran and Silnutzer, 1971; Milton, 1982), fully penetrable cylinders (FPC) (Torquato and Beasley, 1986a; Joslin and Stell, 1986a) and the equilibrium distribution of impenetrable cylinders, Eq. (26)

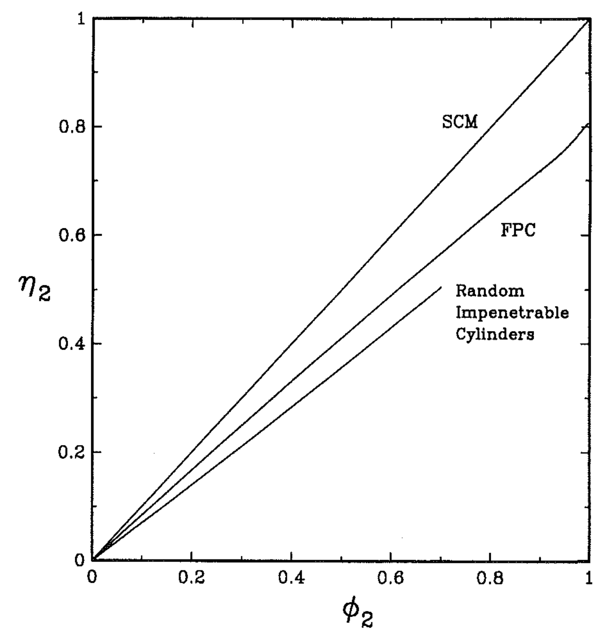

Fig. 2 Three-point parameter η_2, defined by Eq. (2), for random arrays of cylinders, including the symmetric-cell material (SCM) with cylindrical cells (Beran and Silnutzer, 1971; Milton, 1982), fully penetrable cylinders (FPC) (Torquato and Beasley, 1986b; Joslin and Stell, 1986a) and the equilibrium distribution of impenetrable cylinders, Eq. (38)

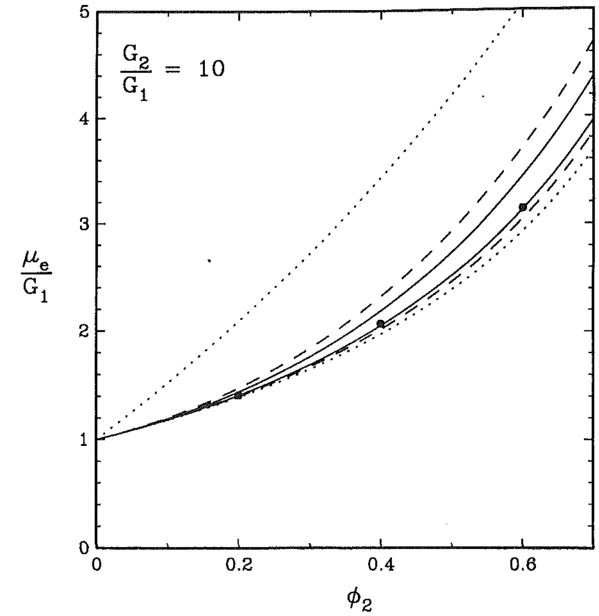

Fig. 3 Bounds on the scaled effective axial shear modulus μ_e/G_1 versus the cylinder volume fraction ϕ_2 at $G_2/G_1 = 10$; ······· second-order bounds (Hashin, 1965; 1970); - - - - third-order bounds (Silnutzer, 1972); and ——— fourth-order bounds (Milton, 1981) for an equilibrium distribution of impenetrable cylinders. The black circles are computer-simulation determinations of the true scaled effective axial shear modulus (Kim and Torquato, 1990).

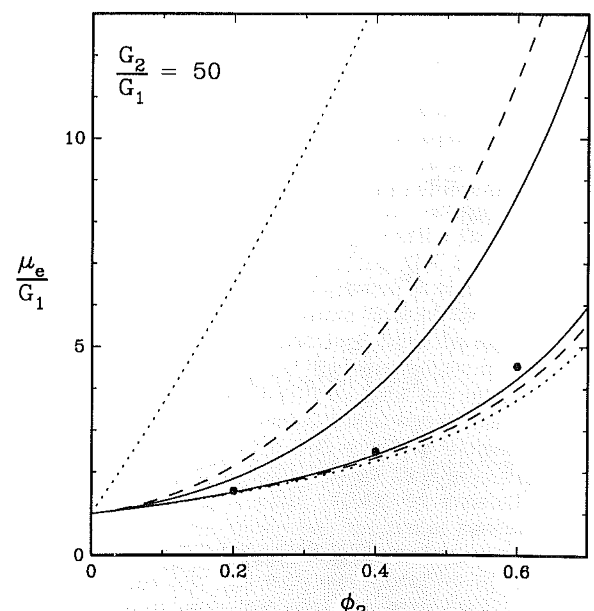

Fig. 4 As in Fig. 3, with $G_2/G_1 = 50$

accurate estimate of the effective parameter (even when the bounds are not tight) since the stiffer phase (fibers) can never form large connected clusters for $0 \leq \phi_2 \leq 0.7$. Equilibrium distributions of cylinders do not form interparticle contacts until the random close-packing volume fraction $\phi_2^c \simeq 0.81$ is reached (Stillinger, DiMarzio and Kornegay, 1964).

In Fig. 3, we plot third and fourth-order bounds on the scaled effective axial shear modulus μ_e/G_1 (Eqs. (3) and (14)) as a function of the cylinder volume fraction ϕ_2 for a composite with a shear modulus ratio $G_2/G_1 = 10$. Included in this figure is the corresponding second-order bounds derived by Hashin and highly accurate Monte Carlo computer simulation data recently obtained by Kim and Torquato (1990) for the same equilibrium impenetrable-cylinder model for the mathematically equivalent effective transverse conductivity. It is seen that the third-order Silnutzer bounds substantially improve upon the second-order bounds; the fourth-order Milton bounds, in turn, are more restrictive than the third-order bounds and indeed are narrow enough to provide a good estimate of μ_e for a wide range of ϕ_2. As expected, most of the improvement comes in the upper bounds. Note that since the superposition approximation evaluation of ζ_2 (Torquato and Lado, 1988a) overestimates it at high ϕ_2, the effect of using the more accurate relation (26) for ζ_2 is to shift the bounds downward towards the second-order lower bound. Observe also the excellent

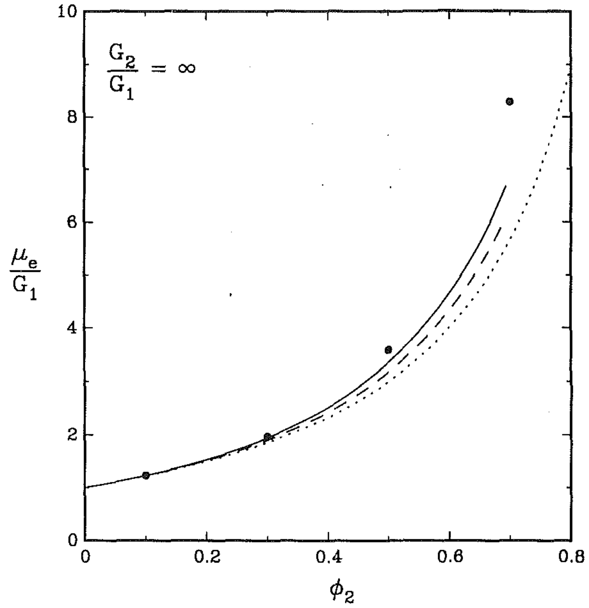

Fig. 5 As in Fig. 3, with $G_2/G_1 = \infty$; upper bounds are not shown since they diverge to infinity in this limit

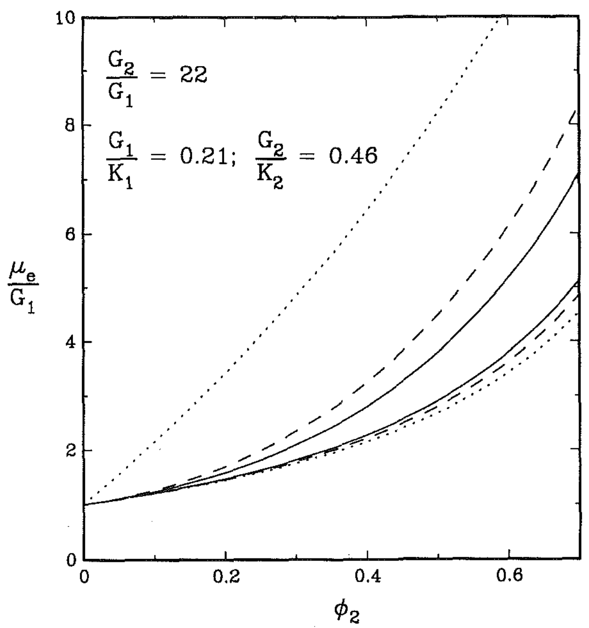

Fig. 6 As in Fig. 3 for a glass-epoxy composite with $G_2/G_1 = 22$, $G_1/K_1 = 0.21$ and $G_2/K_2 = 0.46$. Computer-simulation data are not available in this case.

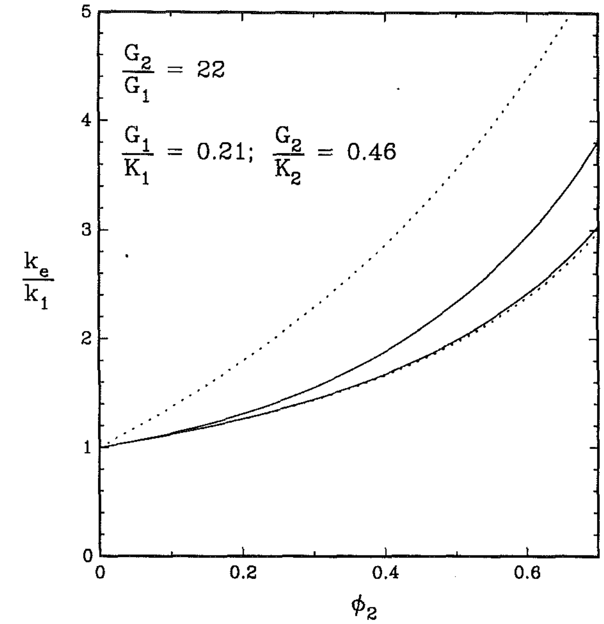

Fig. 7 Bounds on the scaled effective transverse bulk modulus k_e/k_1 versus the cylinder volume fraction ϕ_2 for a glass-epoxy composite for which $G_2/G_1 = 22$, $G_1/K_1 = 0.21$ and $G_2/K_2 = 0.46$; · · · · second-order bounds (Hill, 1964; Hashin, 1965); ——— third-order bounds (Silnutzer, 1972)

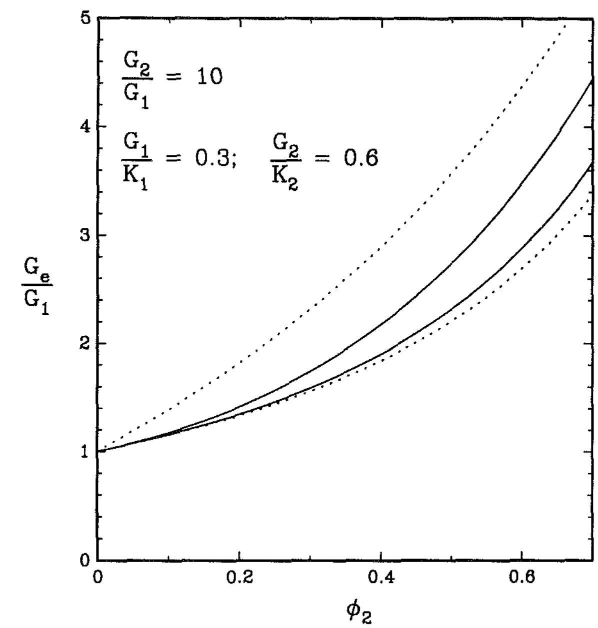

Fig. 8 Bounds on the scaled effective transverse shear modulus G_e/G_1 versus the cylinder volume fraction ϕ_2 for a composite in which $G_2/G_1 = 10$, $G_1/K_1 = 0.3$ and $G_2/K_2 = 0.6$; · · · · second-order bounds (Hill, 1964; Hashin, 1965); ——— third-order bounds (Silnutzer, 1972)

agreement between the computer simulation data and the fourth-order lower bound.

In Fig. 4, we plot the same bounds for the case where the fibers are 50 times more rigid than the matrix ($G_2/G_1 = 50$). Not surprisingly, the bounds widen but, for reasons noted earlier, the fourth-order lower bound still provides an accurate estimate of the effective axial shear modulus for a wide range of cylinder volume fractions.

This latter point is further substantiated by comparing the bounds in the extreme case of super rigid inclusions relative to the matrix ($G_2/G_1 = \infty$) to the computer-simulation data (see Fig. 5). Although all of the upper bounds diverge to infinity here, it is seen that the fourth-order lower bound still provides a good estimate of μ_e for a wide range of ϕ_2.

Figure 6 shows second, third and fourth-order bounds on the scaled effective axial shear modulus μ_e/G_1 for a glass-epoxy composite for which $G_2/G_1 = 22$, $G_1/K_1 = 0.21$ and $G_2/K_2 = 0.46$. Based upon the previous observations, the fourth-order lower bound gives a highly accurate estimate of μ_e for this commonly employed fiber-reinforced material.

Figure 7 depicts the third-order Silnutzer bounds on the scaled effective transverse bulk modulus k_e/k_1 (Eq. (6)) for our model as a function of ϕ_2 for a glass-epoxy composite for which $G_2/G_1 = 22$, $G_1/K_1 = 0.21$, and $G_2/K_2 = 0.46$. Included in the figure are the corresponding second-order bounds (Hill, 1964; Hashin, 1965). The third-order bounds provide significant improvement over the second-order bounds and are tight enough to yield good estimates of k_e for a wide range of cylinder volume fractions.

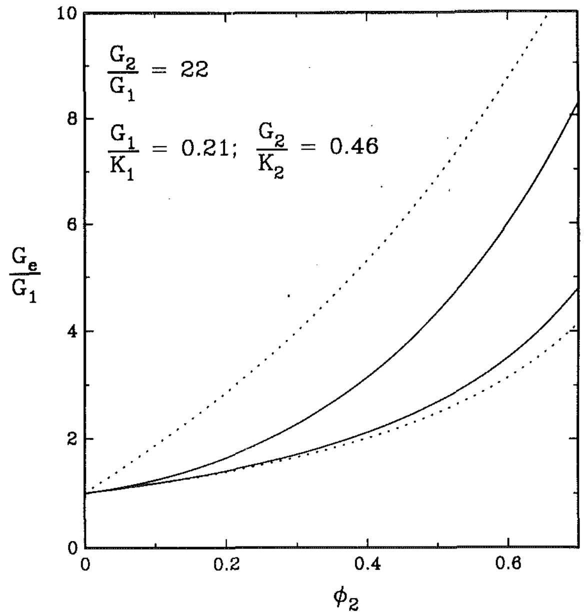

Fig. 9 As in Fig. 8 for a glass-epoxy composite for which $G_2/G_1=22$, $G_1/K_1=0.21$ and $G_2/K_2=0.46$

In Fig. 8, we plot second and third-order bounds on the scaled effective transverse shear modulus G_e/G_1 for our model of a composite with $G_2/G_1=10$, $G_1/K_1=0.3$ and $G_2/K_2=0.6$. Figure 9 shows the same bounds for a glass-epoxy composite with $G_2/G_1=22$, $G_1/K_1=0.21$ and $G_2/K_2=0.46$. Again, the third-order lower bound should yield a good estimate of G_e for a wide range of ϕ_2.

5 Conclusion

Four-point bounds on the effective axial shear modulus and three-point bounds on the effective transverse bulk and shear moduli have been computed for a transversely isotropic, fiber-reinforced material composed of a random array of infinitely long, parallel, impenetrable, circular cylinders for cylinder volume fractions up to 70 percent. This represents the first calculation of such bounds of the transverse bulk and shear moduli for this practically useful model. We exploited a key property of the microstructural parameters ζ_2 and η_2 to accurately compute them for fixed volume fraction ϕ_2, namely, for a large class of random distributions of oriented cylinders (be they overlapping or possessing size distribution), the low-volume fraction expansions of the parameters (which are easily determined) are excellent approximations over a wide range of ϕ_2 (Torquato, 1991). A striking result is that the lower bounds provide remarkably accurate estimates of the elastic moduli, even when the cylinders are much more rigid than the matrix.

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