Effective energy of nonlinear elastic and conducting composites: Approximations and cross-property bounds

Leonid Gibiansky[a] and Salvatore Torquato[b]
Princeton Materials Institute and Department of Civil Engineering and Operations Research, Princeton University, Princeton, New Jersey, 08540

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By using similar techniques that were employed in an earlier article on nonlinear conductivity [L. V. Gibiansky and S. Torquato, J. Appl. Phys. **84**, 301 1998], we find approximations for the effective energy of nonlinear, isotropic, elastic dispersions in arbitrary space dimension d. We apply our results for incompressible dispersions with rigid or liquid inclusions and, more generally, with a power-law-type shear energy. It is shown that the new approximations lie within the best available rigorous upper and lower bounds on the effective energy. We also develop bounds on the effective energy of nonlinear conducting media with voids or cracks, purely in terms of the effective and phase elastic moduli of the media.
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I. INTRODUCTION

This article is a natural continuation of our previous work[1] where we obtained new approximations on the effective energy of nonlinear, isotropic, conducting dispersions. Here, we extend the investigation to treat the effective properties of nonlinear, isotropic, elastic composites made of two isotropic phases, and obtain additional results for the nonlinear conductivity problem.

Work on bounding the effective moduli of linear elastic composites began with Hill[2] who proved arithmetic- and harmonic-mean bounds on the effective properties. Hashin and Shtrikman[3,4] found the best possible upper and lower bounds on the effective bulk and shear moduli given only volume fraction information. Coupled bulk–shear moduli bounds[5–7] improved upon the Hashin and Shtrikman[3,4] results by taking into account the interdependence of the bulk and shear moduli. By incorporating higher-order microstructural information, one can further improve upon the linear Hashin–Shtrikman bounds.[5,6,8–13] On the other hand, cross-property conductivity–elastic moduli bounds[11,6,14–16] relate the effective conductivity and effective elastic moduli of two-phase isotropic composites, and are valid even in the limit of cracked media.[17]

Bounding the effective properties of nonlinear composites is a much more difficult problem. Talbot and Willis[18] and Willis[19] suggested generalizations of the Hashin–Shtrikman variational method to treat nonlinear composites. Talbot and Willis[20] used the new method to compute bounds on the effective properties of nonlinear heterogeneous dielectrics and compared them with self-consistent estimates. Ponte Castaneda[21,22] introduced a method that allows one to bound or approximate the effective properties of a nonlinear composite by using a bound or an approximation for the effective properties of a comparison composite with an identical microstructure but with linear constitutive relations. A similar method for composites with a power-law energy was developed by Suquet.[23,24] A comprehensive review of the literature can be found in the recent review by Ponte-Castaneda and Suquet.[25] New advances on nonlinear bounds have been made by Willis[26] and Talbot and Willis.[27]

In this article, we consider two problems. In Sec. II, we develop approximations for the effective shear modulus of nonlinear incompressible dispersions. This is done by applying an approach of Ponte Castaneda[21,22] that requires knowledge of the effective shear modulus of a linear comparison material. Here, we use a recent analytical approximation found by Torquato[28] for the effective shear modulus of a linear material. This expression turns out to be useful in approximating the linear effective shear modulus of dispersions. We study particular examples such as incompressible dispersions with rigid or liquid inclusions, and two-phase composites with the phases characterized by a power-law dependence of the phase energy on the applied shear field.

In Sec. III, we develop cross-property bounds on the effective conductivity of an isotropic nonlinear conductor with voids or cracks by using cross-property bounds[17] on the conductivity of a linear composite with voids or cracks and known effective bulk or shear modulus. We test our inequalities for two specific examples. First, we compare the approximation[1] for the effective energy of a power-law *porous conductor* and the corresponding cross-property bound based on the Torquato[28] approximation for the effective bulk modulus of the same composite. Then, we compare our cross-property bound on the effective energy of a power-law *conductor with cracks* to the noninteracting cracks approximation for a linear comparison material. In Sec. IV, we make concluding remarks.

II. EFFECTIVE ENERGY OF AN INCOMPRESSIBLE COMPOSITE

A. General approximation

As we will see in this section, an approach developed in the earlier paper[1] to study the nonlinear conductivity prob-

[a]Electronic mail: gibiansky@erols.com
[b]Electronic mail: torquato@material.princeton.edu

lem allows for straightforward generalization to the problem of nonlinear elastic composites. Specifically, consider an isotropic physically nonlinear *incompressible* elastic material. Following Ponte Castaneda and Suquet,[25] we assume that the complementary energy density u has a form

$$u(\boldsymbol{\tau})=\psi(\tau_{\rm eq}), \tag{1}$$

where $\boldsymbol{\tau}$ is the stress tensor, $\psi(\tau_{\rm eq})$ is the function of the scalar variable that describes the shear part of the complementary energy. Here,

$$\tau_{\rm eq}=\sqrt{\frac{d}{2}\,\boldsymbol{\tau}_e:\boldsymbol{\tau}_e} \tag{2}$$

is the Von Mises equivalent stress with

$$\boldsymbol{\tau}_e=\boldsymbol{\tau}-\frac{1}{d}\,{\rm Tr}[\boldsymbol{\tau}]\mathbf{I}, \tag{3}$$

being the deviatoric (trace-free, ${\rm Tr}[\boldsymbol{\tau}_e]=0$) stress, and d being the spatial dimension.

There are two examples of constitutive potentials that are of special interest.[25] One is important for the J_2-deformation theory of plasticity in which

$$\psi(\tau_{\rm eq})=\frac{1}{6\mu_0}\tau_{\rm eq}^2+\frac{\tau_0\epsilon_0}{n+1}\left(\frac{\tau_{\rm eq}}{\tau_0}\right)^{n+1}, \quad (d=3), \tag{4}$$

where μ_0 is the initial elastic shear modulus, τ_0 is a reference stress, ϵ_0 is a reference strain, and $n\geqslant 1$ is the hardening exponent. The other potential appropriate for high-temperature creep of metals can be modeled by

$$\psi(\tau_{\rm eq})=\frac{\tau_0\epsilon_0}{n+1}\left(\frac{\tau_{\rm eq}}{\tau_0}\right)^{n+1}, \tag{5}$$

where τ_0 is a reference stress, ϵ_0 is a reference strain rate, and n is the power exponent. Note that a Newtonian viscous material corresponds to $n=1$, where $\mu=\tau_0/3$ denotes the viscosity. The Von Mises rigid ideally plastic material corresponds to the limit $n=\infty$, where τ_0 denotes the flow stress in tension.

For linear materials, the shear part of the energy is quadratic, i.e.,

$$\psi(\tau_{\rm eq})=\frac{1}{2d\mu}\tau_{\rm eq}^2, \tag{6}$$

in which the shear modulus μ is independent of the applied field.

For an isotropic composite with the periodic cell Ω made of two isotropic phases, the local complementary energy density function $\Psi(\mathbf{x},\tau_{\rm eq})$ has the form

$$\Psi(\mathbf{x},\tau_{\rm eq})=\chi_1(\mathbf{x})\psi_1(\tau_{\rm eq})+\chi_2(\mathbf{x})\psi_2(\tau_{\rm eq}), \tag{7}$$

where $\chi_1(\boldsymbol{x})$ and $\chi_2(\boldsymbol{x})$ are the characteristic functions of the regions occupied by phase 1 and phase 2, respectively, and $\boldsymbol{x}$ is the position vector. We assume that $\psi_i(\tau_{\rm eq})$ $(i=1,2)$ are continuous and convex functions such that

$$\psi_i(\tau_{\rm eq})\geqslant 0\ \forall\tau_{\rm eq}, \quad \psi_i(0)=0, \quad i=1,2. \tag{8}$$

Then, the effective complementary energy potential is given by

$$\hat{\Psi}(\overline{\tau}_{\rm eq})=\inf_{\substack{\Omega\text{-periodic }\ \boldsymbol{\tau}(\mathbf{x})\\ \nabla\cdot\boldsymbol{\tau}(\mathbf{x})=0\\ \langle\boldsymbol{\tau}(\mathbf{x})\rangle=\overline{\boldsymbol{\tau}}}}\langle\Psi(\mathbf{x},\tau_{\rm eq})\rangle, \tag{9}$$

where angular brackets denote the volume integral of the bracketed quantity over the periodic cell Ω. Our main goal is to find bounds or approximations for the effective complementary energy (9).

Note that formally the problem for nonlinear incompressible materials is completely analogous to the nonlinear conductivity problem where the phase conductivity constants are replaced by the inverse phase shear moduli, and the magnitude of the electrical field is replaced by the Von Mises equivalent stress. For example, the following theorem[22] is a direct analogue of the corresponding result used in our previous paper:[1]

Theorem 1: The effective complementary energy function $\hat{\Psi}(\overline{\tau}_{\rm eq})$ of the nonlinear incompressible elastic composite satisfies the inequality

$$\hat{\Psi}(\overline{\tau}_{\rm eq})\geqslant\max_{\mu_i^0>0,\ i=1,2}\left\{\hat{\Psi}^0(\overline{\tau}_{\rm eq})-\sum_{i=1}^{2}\phi_i v_i(\mu_i^0)\right\}, \tag{10}$$

where $\hat{\Psi}^0(\overline{\tau}_{\rm eq})$ is the effective complementary energy function of a linear comparison incompressible composite with the shear moduli μ_i^0, ϕ_i are the volume fractions of phases, and the functions $v_i(\mu_i^0)$ are given by the relations

$$v_i(\mu_i^0)=\sup_{\tau_{\rm eq}}\left\{\frac{1}{2d\mu_i^0}\,\overline{\tau}_{\rm eq}^2-\psi_i(\tau_{\rm eq})\right\}, \quad i=1,2. \tag{11}$$

Similar to the conductivity problem,[1] we assume that the linear *incompressible* comparison composite is approximated (or bounded) by the expression

$$\hat{\Psi}^0(\overline{\tau}_{\rm eq})=\frac{1}{2d\mu_e^0}\,\overline{\tau}_{\rm eq}^2, \tag{12}$$

where

$$(\mu_e^0)^{-1}=\phi_1(\mu_1^0)^{-1}+\phi_2(\mu_2^0)^{-1}-\frac{\phi_1\phi_2[(\mu_1^0)^{-1}-(\mu_2^0)^{-1}]^2}{\phi_2(\mu_1^0)^{-1}+\phi_1(\mu_2^0)^{-1}+2Y_0/d}. \tag{13}$$

Here, Y_0 may be either constant or be a function of the type

$$Y_0=\eta_1(\mu_1^0)^{-1}+\eta_2(\mu_2^0)^{-1}-\frac{\eta_1\eta_2[(\mu_1^0)^{-1}-(\mu_2^0)^{-1}]^2}{\eta_2(\mu_1^0)^{-1}+\eta_1(\mu_2^0)^{-1}+dZ_0/2}, \tag{14}$$

with $\eta_1=(1-\eta_2)\in[0,1]$ being some fixed parameter, and Z_0 being yet another constant. Many known bounds on the effective properties of two-phase linear incompressible elastic composites can be presented in form (12)–(14). In particular, the Hashin–Shtrikman shear modulus bounds[3,4] are given by Eq. (13) where $Y_0=1/\mu_1$ or $Y_0=1/\mu_2$, and may be obtained from Eqs. (12)–(14) by assigning the values $\eta_1=1$ $(\eta_2=0)$ or $\eta_1=0$ $(\eta_2=1)$. The Silnutzer three-point bounds[12] for the two-dimensional ($d=2$) incompressible elastic composite are given by expressions (13) and (14) with $Z_0=0$ and $Z_0=\infty$, and $\eta_1=1-\eta_2$ being the three-point geometrical parameters.[11–12] Finally, the Milton–Phan-Thien three-point upper shear modulus bound[5] in three dimensions ($d=3$) is given by Eqs. (13) and (14), where $Z_0=0$. Note that the Milton–Phan-Thien three-point lower shear modulus bound[5] can be presented in form (13) and (14) only if the shear modulus of one of the phases (say, phase 2) is infinite. Then, it is given by Eq. (13) with

$$Y_0=Y_M=\frac{(21\eta_1-5\zeta_1)\zeta_1}{11\zeta_1+5\eta_1}, \tag{15}$$

where $\zeta_1=1-\zeta_2$ are the other three-point geometrical parameters.[10–12,29]

Torquato[28] derived approximations for the bulk and shear moduli of dispersions in terms of the three-point parameters ζ_2 and η_2. In the special case of incompressible composites, his formula for the effective shear modulus of the d-dimensional dispersions (where phase 1 and phase 2 are the matrix and dispersed phases, respectively) is described by formulas (13) and (14), where

$$Z_0=B/\mu_1, \tag{16}$$

and B is a parameter, independent of the phase properties, given by

$$B=\frac{2(2-d\eta_2)}{d(d-2\eta_2)}. \tag{17}$$

The Torquato approximation [Eqs. (13)–(17)] is accurate for a wide range of phase volume fractions and elastic moduli provided that the particles do not form large clusters.

Another advantage of a linear comparison material with effective properties specified by Eqs. (13) and (14) is that in this case the bound (10) can be greatly simplified by using the procedure developed by Ponte Castaneda.[22] Specifically, this method yields the following approximation for the effective energy $\hat{\Psi}(\overline{\tau}_{eq})$ of nonlinear isotropic dispersions:

$$\hat{\Psi}(\overline{\tau}_{eq})=\min_{\omega,\gamma}\{\phi_2\psi_2[\overline{\tau}_{eq}\sqrt{(1+\phi_1\omega)^2+2\phi_1\eta_2\omega^2(1+\eta_1\gamma)^2/d}]$$
$$+\phi_1\psi_1[\overline{\tau}_{eq}\sqrt{(1-\phi_2\omega)^2+2\phi_2\eta_1\omega^2(1-\eta_2\gamma)^2/d+B\phi_2\eta_1\eta_2\omega^2\gamma^2}]\}. \tag{18}$$

Here, we need to perform an optimization over only two scalar parameters $\omega\in(-\infty,\infty)$ and $\gamma\in(-\infty,\infty)$. This can be done either analytically (if the energy functions of the nonlinear phases are sufficiently simple), or numerically.

The approximation based on the Torquato[28] formula is given by Eq. (18) with the parameter B as in Eq. (17). This is valid for a composite of any dimensionality d. The Silnutzer three-point bounds[12] for a two-dimensional ($d=2$) linear comparison composite result in the nonlinear approximations

$$\hat{\Psi}(\overline{\tau}_{eq})=\min_{\omega,\gamma}\{\phi_2\psi_2[\overline{\tau}_{eq}\sqrt{(1+\phi_1\omega)^2+2\phi_1\eta_2\omega^2(1+\eta_1\gamma)^2/d}]+\phi_1\psi_1[\overline{\tau}_{eq}\sqrt{(1-\phi_2\omega)^2+2\phi_2\eta_1\omega^2(1-\eta_2\gamma)^2/d}]\} \tag{19}$$

and

$$\hat{\Psi}(\overline{\tau}_{eq})=\min_{\omega}\{\phi_2\psi_2[\overline{\tau}_{eq}\sqrt{(1+\phi_1\omega)^2+2\phi_1\eta_2\omega^2/d}]$$
$$+\phi_1\psi_1[\overline{\tau}_{eq}\sqrt{(1-\phi_2\omega)^2+2\phi_2\eta_1\omega^2/d}]\}. \tag{20}$$

The Milton–Phan-Thien three-point upper shear modulus bound[5] in three dimensions ($d=3$) allows us to derive the bound (19) for the effective complementary energy of a nonlinear composite. Finally, using the Hashin–Shtrikman shear modulus bounds[3,4] for the effective shear modulus of a linear comparison material one arrives at the expressions

$$\hat{\Psi}(\overline{\tau}_{eq})=\min_{\omega}\{\phi_2\psi_2[\overline{\tau}_{eq}\sqrt{(1+\phi_1\omega)^2}]$$
$$+\phi_1\psi_1[\overline{\tau}_{eq}\sqrt{(1-\phi_2\omega)^2+2\phi_2\omega^2/d}]\}, \tag{21}$$

$$\hat{\Psi}(\overline{\tau}_{eq})=\min_{\omega}\{\phi_2\psi_2[\overline{\tau}_{eq}\sqrt{(1+\phi_1\omega)^2+2\phi_1\omega^2/d}]$$
$$+\phi_1\psi_1[\overline{\tau}_{eq}\sqrt{(1-\phi_2\omega)^2}]\}. \tag{22}$$

One cannot proceed further without specifying the phase energy functions. In what follows, we consider several specific examples.

B. Incompressible liquid inclusions

Let us assume that the dispersed phase 2 is an incompressible liquid, i.e.,

$$\psi_2(\tau_{eq})=\begin{cases}0, & \text{if } \tau_{eq}=0;\\ \infty, & \text{if } \tau_{eq}\neq 0.\end{cases} \tag{23}$$

In such a case, the right-hand side of Eq. (18) is equal to infinity unless the argument of the function ψ_2 is equal to zero, i.e.,

$$\overline{\tau}_{eq}\sqrt{(1+\phi_1\omega)^2+2\phi_1\eta_2\omega^2(1+\eta_1\gamma)^2/d}=0. \tag{24}$$

This defines the optimal values of the parameters ω and γ as

$$\omega=-1/\phi_1,\quad \gamma=-1/\eta_1. \tag{25}$$

Then, the energy of the composite is approximated by the expression

$$\hat{\Psi}(\overline{\tau}_{eq})=\phi_1\psi_1(\overline{\tau}_{eq}\sqrt{\alpha_L}), \tag{26}$$

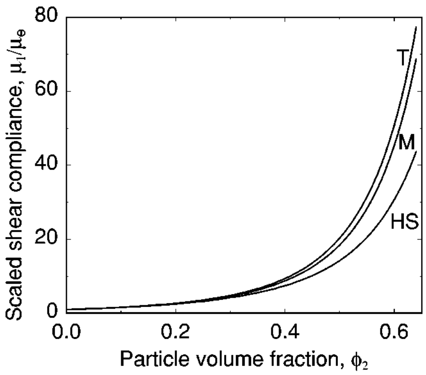

FIG. 1. Scaled effective shear compliance μ_1/μ_e vs the particle volume fraction ϕ_2 for random equilibrium arrays of spherical incompressible liquid inclusions in an incompressible matrix with a quartic energy (30). Here, T is the approximation (18) based on the Torquato (Ref. 28) formula, M is the three-point bound (19) based on the Milton–Phan-Thien (Ref. 5) linear bound, and HS is the two-point bound (21) based on the Hashin–Shtrikman (Ref. 3) bound for the linear comparison material.

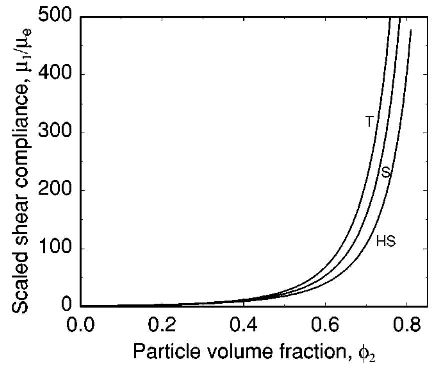

FIG. 2. Scaled effective shear compliance μ_1/μ_e vs the particle volume fraction ϕ_2 for random equilibrium arrays of disk-like incompressible liquid inclusions in an incompressible matrix with a quartic energy (30). Here, T is the approximation (18) based on the Torquato (Ref. 28) formula, S is the three-point bound (19) based on the Silnutzer (Ref. 12) linear bound, and HS is the two-point bound (21) based on the Hashin (Ref. 4) bound for the linear comparison material.

where the parameter α_L is given by

$$\alpha_L = \frac{\eta_1 + 2\phi_2/d + B\phi_2\eta_2}{\eta_1\phi_1^2}. \tag{27}$$

For the Torquato approximation (18), the parameter B is given by Eq. (17), for the Milton–Phan-Thien $(d=3)$ or Silnutzer $(d=2)$ approximation (19) we have $B=0$, and for the Hashin–Shtrikman approximation (21) we take $\eta_1=1$, $(\eta_2=0)$, resulting in $\alpha_L=(1+2\phi_2/d)/\phi_1^2$.

In particular, for a matrix made of a material with the energy function (4), the effective energy is equal to

$$\hat{\Psi}(\overline{\tau}_{eq}) = \frac{\phi_1\alpha_L}{6\mu_1}\overline{\tau}_{eq}^2 + \frac{f_1\tau_0\epsilon_0}{n+1}\left(\frac{\overline{\tau}_{eq}\sqrt{\alpha_L}}{\tau_0}\right)^{n+1}. \tag{28}$$

For high-temperature creep (5), the effective potential is given by

$$\Psi(\overline{\tau}_{eq}) = \frac{\phi_1\tau_0\epsilon_0}{n+1}\left(\frac{\overline{\tau}_{eq}\sqrt{\alpha_L}}{\tau_0}\right)^{n+1}. \tag{29}$$

Let us illustrate our results for a composite made of a matrix with the power-law shear energy

$$\psi_1(\tau_{eq}) = \frac{1}{d(n+1)\mu_1}\tau_{eq}^{(n+1)}, \tag{30}$$

and the effective energy

$$\hat{\Psi}(\tau_{eq}) = \frac{1}{d(n+1)\mu_e}\overline{\tau}_{eq}^{(n+1)}, \tag{31}$$

where

$$\frac{\mu_1}{\mu_e} = \phi_1\alpha_L^{(n+1)/2}. \tag{32}$$

Figure 1 compares our approximation for the scaled compliance μ_1/μ_e with rigorous bounds for dispersions consisting of a random equilibrium array of nonoverlapping spherical liquid inclusions in a matrix material given by Eq. (30) ($d=3$, $n=3$). Such composites are isotropic by construction, and the geometrical parameters ζ_2 and η_2 can be expressed as a function of the volume fraction as follows:[30]

$$\begin{aligned}\zeta_2 &= 0.21068\phi_2 - 0.04693\phi_2^2,\\ \eta_2 &= 0.48274\phi_2, \quad \phi_2\in[0,\ 0.64] \quad (d=3).\end{aligned} \tag{33}$$

One can see that approximation (18) (curve T) lies above Eq. (19) based on the Milton–Phan-Thien shear modulus bound (curve M). For purposes of comparison, we also computed approximation (21) based on the Hashin–Shtrikman shear modulus bound (curve HS). Note that approximation (22) diverges to infinity in this case. We see that the approximation based on the Torquato[28] formula satisfies the rigorous two- and three-point bounds.

Similar results for two-dimensional dispersions $(d=2)$ consisting of a random equilibrium array of nonoverlapping disk-like liquid inclusions in a matrix material (30) $(n=3)$ are shown in Fig. 2. We assume that the geometrical parameters are given by[30]

$$\begin{aligned}\zeta_2 &= 1/3\phi_2 - 0.05707\phi_2^2,\\ \eta_2 &= 56/81\phi_2 + 0.0428\phi_2^2, \quad \phi_2\in[0,\ 0.81] \quad (d=2).\end{aligned} \tag{34}$$

Here, we computed the Silnutzer bound (19) for $d=2$ instead of the analogous Milton–Phan-Thien bound (19) with $d=3$. We see that in the two-dimensional case, the approximation based on the Torquato[28] formula also satisfies rigorous two- and three-point bounds.

C. Rigid inclusions

Let us now evaluate the estimates (18) on the effective energy of a dispersion with the same structure and matrix phase but with perfectly rigid inclusions, i.e.,

$$\psi_2(\tau_{eq}) = 0, \quad \forall\tau_{eq}. \tag{35}$$

In this case, the optimal values of the parameters ω and γ are those that minimize the argument

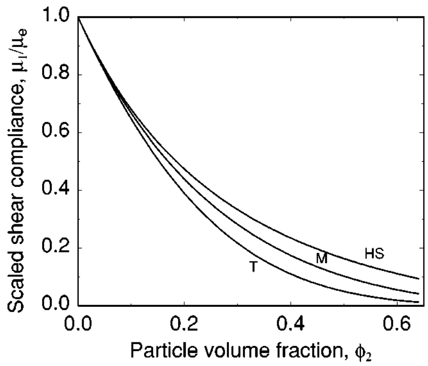

FIG. 3. Scaled effective shear compliance μ_1/μ_e vs the particle volume fraction ϕ_2 for random equilibrium arrays of rigid spherical inclusions for materials with the quartic energy (30), $n=3$. Here, T is the approximation (18) based on the Torquato (Ref. 28) formula, M is the three-point bound (18), (40) based on the Milton–Phan-Thien (Ref. 5) linear bound, and HS is the two-point bound (21) based on the Hashin–Shtrikman (Ref. 3) bound for the linear comparison material.

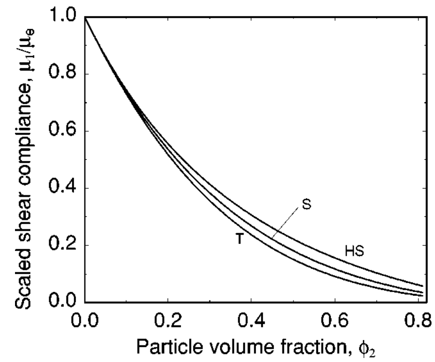

FIG. 4. Scaled effective shear compliance μ_1/μ_e vs the particle volume fraction ϕ_2 for random equilibrium arrays of rigid disk-like inclusions for materials with the quartic energy (30), $n=3$. Here, T is the approximation (18) based on the Torquato (Ref. 28) formula, S is the three-point bound (20) based on the Silnutzer (Ref. 12) linear bound, and HS is the two-point bound (21) based on the Hashin (Ref. 4) bound for the linear comparison material.

$$\overline{\tau}_{\rm eq}\sqrt{(1-\phi_2\omega)^2+2\phi_2\eta_1\omega^2(1-\eta_2\gamma)^2/d+B\phi_2\eta_1\eta_2\omega^2\gamma^2}, \tag{36}$$

of the function ψ_1 in Eq. (18). They are equal to

$$\omega=\frac{\eta_2+dB/2}{\phi_2\eta_2+B\eta_1+dB\phi_2/2}, \quad \gamma=\frac{1}{\eta_2+dB/2}. \tag{37}$$

Then, the energy of the composite is approximated by the expression

$$\hat{\Psi}(\overline{\tau}_{\rm eq})=\phi_1\psi_1(\overline{\tau}_{\rm eq}\sqrt{\alpha_R}), \tag{38}$$

where

$$\alpha_R=\frac{B\eta_1}{\eta_2\phi_2+B\eta_1+dB\phi_2/2}. \tag{39}$$

For the Torquato approximation (18), the parameter B is given by Eq. (17). For the Silnutzer approximation (20), we have that $B=\infty$, resulting in $\alpha_R=\eta_1/(\eta_1+d\phi_2/2)$. In the case of the nonlinear approximation based on the Milton–Phan-Thien bound (13), (15) we have

$$\alpha_R=\alpha_R^M=\frac{2Y_M}{2Y_M\eta_1+d\phi_2}. \tag{40}$$

Finally, for the Hashin–Shtrikman approximation (21) we should assign $\eta_1=1$, $(\eta_2=0)$, resulting in

$$\alpha_R=2/(2+d\phi_2).$$

For composites with a matrix phase potential given by Eq. (4), (5) or (30), the effective energy is given by expressions (28), (29), and (31), where the parameter α_L is replaced by the corresponding parameter α_R.

Figure 3 compares our approximation with the other estimates for a power-law material (30) with $n=3$, reinforced by an equilibrium array of rigid spherical inclusions. One can see that approximation (18) (curve T) lies below the three-point approximation (40) of Milton–Phan-Thien (curve M), and the Hashin–Shtrikman approximation (21) (curve HS). The other Hashin–Shtrikman approximation (22) degenerates to zero in this case.

Figure 4 depicts similar results for $d=2$, a power-law material specified by Eq. (30) with $n=3$ reinforced by an equilibrium array of rigid disks. Approximation (18) (curve T) lies below the three-point approximation (20) of Silnutzer (curve S), and the Hashin–Shtrikman approximation (21) (curve HS). The other Hashin–Shtrikman approximation (22) degenerates to zero. The approximation based on the Torquato[28] formula lies below the approximations based on the two- and three-point bounds.

D. Two phases with a power-law energy

Now we turn our attention to the more general problem of a two-phase incompressible composite with finite phase shear moduli. We evaluate expression (18) for the effective energy of the composite with power-law phase energies (30) with two different shear moduli μ_1 and μ_2. The effective energy has the same power-law behavior (31) with the approximation for the scaled compliance μ_1/μ_e given by

$$\frac{\mu_1}{\mu_e}=\min_{\omega,\gamma}\Big\{\phi_2\frac{\mu_1}{\mu_2}[(1+\phi_1\omega)^2+2\phi_1\eta_2\omega^2(1+\eta_1\gamma)^2/d]^{(n+1)/2}+\phi_1[(1-\phi_2\omega)^2+2\phi_2\eta_1\omega^2(1-\eta_2\gamma)^2/d+B\phi_2\eta_1\eta_2\omega^2\gamma^2]^{(n+1)/2}\Big\}. \tag{41}$$

However, in this case, the optimal values of the parameters ω and γ cannot be found analytically, and therefore, we find them numerically. Figure 5 gives the dependence of the scaled compliance μ_1/μ_e on the volume fraction of phase 2 for $d=3$, the phase contrast ratio

$$\mu_1/\mu_2=10, \tag{42}$$

and $n=1$, $n=3$, and $n=9$. Corresponding plots for the phase contrast ratio

$$\mu_1/\mu_2=0.1, \tag{43}$$

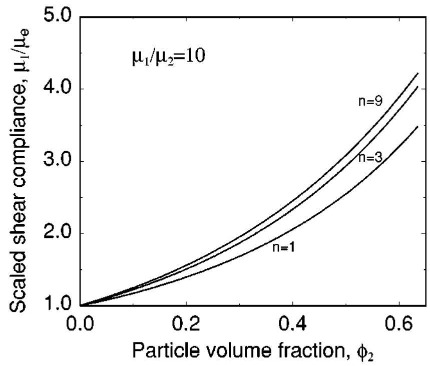

FIG. 5. Scaled effective shear compliance μ_1/μ_e vs the particle volume fraction ϕ_2 for random equilibrium arrays of spherical inclusions for materials with a power-law energy and $\mu_1/\mu_2=10$ for several values of the exponent as obtained from Eq. (18).

are shown in Fig. 6. In this case, the curves corresponding to the exponents $n \geqslant 3$ coincide in the scale of Fig. 6.

The scaled shear compliance μ_1/μ_e increases with the exponent n if the dispersed particles are more compliant than the matrix, and decreases with n if the dispersed particles are stiffer than that of the matrix.

III. CROSS-PROPERTY BOUNDS FOR THE ENERGY OF POROUS OR CRACKED NONLINEAR CONDUCTORS

In this section we use cross-property conductivity–elastic moduli bounds[15–17] for linear materials, and the Ponte Castaneda procedure[21,22] to obtain bounds on the effective energy of two-phase nonlinear conductors. We assume that one can measure or estimate the linear bulk moduli κ_i and the shear moduli μ_i of the phases, and the effective bulk modulus κ_e and/or the shear modulus μ_e of the composite.

Consider an isotropic two-phase conducting composite which is characterized by the local energy density function

$$W(\boldsymbol{x},E)=\sum_{i=1}^{2} \chi_i(\mathbf{x})w_i(E), \quad w_i(E)\geqslant 0 \quad \forall E, \qquad w_i(0)=0, \quad i=1,2, \tag{44}$$

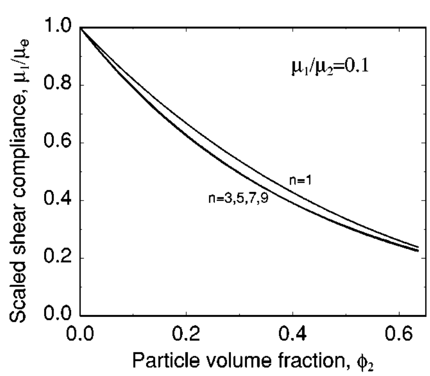

FIG. 6. Scaled effective shear compliance μ_1/μ_e vs the particle volume fraction ϕ_2 for random equilibrium arrays of spherical inclusions for materials with a power-law energy and $\mu_1/\mu_2=0.1$ for several values of the exponent as obtained from Eq. (18). Curves corresponding to the powers $n=3,5,7,9$ coincide in the scale of the figure.

where E is the magnitude of electrical field $\mathbf{E}=-\nabla\varphi$ (φ is the electric potential). For linear conductors, the energy is quadratic and the current is proportional to the applied field, i.e.,

$$w_i^0(E)=\frac{1}{2}\sigma_i^0 E^2, \quad \mathbf{J}=\frac{\partial w_i^0(E)}{\partial \mathbf{E}}=\sigma_i^0\mathbf{E}, \tag{45}$$

where the conductivity constants σ_i^0 are independent of the applied field. For power-law conductors, the energy and the current are given by

$$w_i(E)=\frac{1}{n+1}\sigma_i E^{n+1}, \quad \mathbf{J}=\frac{\partial w^0(E)}{\partial \mathbf{E}}=\sigma_i E^{n-1}\mathbf{E}. \tag{46}$$

The effective energy of the composite is given by

$$\hat{W}(\overline{\mathbf{E}})=\inf_{\substack{\Omega\text{-periodic } \mathbf{E}(\mathbf{x})\\ \nabla\times\mathbf{E}(\mathbf{x})=0\\ \langle\mathbf{E}(\mathbf{x})\rangle=\overline{\mathbf{E}}}} \langle W(\mathbf{x},E)\rangle, \tag{47}$$

where Ω is the periodic cell of the composite. For power-law conductors, the effective energy is given by

$$\hat{W}(\overline{E})=\frac{1}{n+1}\sigma_e E^{n+1}, \tag{48}$$

where σ_e is the effective conductivity constant.

Ponte Castaneda[21,22] proved the following theorem:

Theorem 2: The effective energy function $\hat{W}(\overline{\mathbf{E}})$ of the nonlinear conductor satisfies the inequality

$$\hat{W}(\overline{\mathbf{E}})\geqslant \max_{\sigma_i^0>0,\ i=1,2}\left\{\hat{W}^0(\overline{\mathbf{E}})-\sum_{i=1}^{2}\phi_i v_i(\sigma_i^0)\right\}, \tag{49}$$

where $\hat{W}^0(\overline{\mathbf{E}})$ is the effective energy function of a linear comparison composite with phase conductivities σ_i^0, ϕ_i are the volume fractions of phases, and the functions $v_i(\sigma_i^0)$ are given by the relations

$$v_i(\sigma_i^0)=\sup_E\left\{\frac{1}{2}\sigma_i^0E^2-w_i(E)\right\}, \quad i=1,2. \tag{50}$$

In our earlier papers[14–17] we found bounds on the effective conductivity of linear conducting composites in terms of the effective linear elastic moduli. These bounds can be used to find corresponding bounds on the effective conductivity of the linear comparison material, and thus, on the effective energy of the nonlinear conductor. Cross-property bounds have an especially simple form for composites with voids or cracks, when the conductivity and elastic moduli of the "void phase" 2 are small compared to the corresponding moduli of the matrix phase 1. For a three-dimensional composite, they have the form[17]

$$\sigma_e\geqslant A_\kappa\sigma_1, \quad A_\kappa=\frac{3\kappa_e\kappa_1}{3\kappa_1\kappa_e+2\mu_1(\kappa_1-\kappa_e)M}, \qquad M=\max\left\{1, \frac{1+\nu_1}{1-\nu_1}\right\} \quad (d=3), \tag{51}$$

where κ_e is the effective bulk modulus of the composite, and $\nu_1=(3\kappa_1-2\mu_1)/(6\kappa_1+2\mu_1)$ is the Poisson's ratio of the solid phase. For a two-dimensional composite. For a two-dimensional composite, similar bounds[17] can be expressed as

$$\sigma_e \geqslant A_k \sigma_1, \quad A_k = \frac{k_e(k_1+\mu_1)}{2k_1\mu_1+k_e(k_1-\mu_1)} \quad (d=2), \tag{52}$$

$$\sigma_e \geqslant A_\mu \sigma_1, \quad A_\mu = \frac{\mu_e(k_1+\mu_1)}{\mu_1(k_1+\mu_e)} \quad (d=2), \tag{53}$$

$$\sigma_e \geqslant A_E \sigma_1, \quad A_E = \frac{3E_e}{2E_1+E_e} \quad (d=2), \tag{54}$$

where $k_1=\kappa_1+\mu_1/3$ and $E_1=4k_1\mu_1/(k_1+\mu_1)$ are the two-dimensional bulk and Young's moduli of the solid phase, and k_e, μ_e, and $E_e=4k_e\mu_e/(k_e+\mu_e)$ are the effective bulk, shear, and Young's moduli of the two-dimensional composite. Note that the commonly used notation for the Young's modulus is the same as that for the magnitude of the electrical field. This should not cause any confusion since it is obvious which of them are used from the context of the problem.

Let us first apply the Ponte Castaneda bound (49) to the three-dimensional problem, and use the bound (51) for the effective conductivity of a linear comparison material. This results in the inequality

$$\hat{W}(\overline{\mathbf{E}}) \geqslant \phi_1 \max_{\sigma>0}\left[\frac{1}{2}\frac{A_\kappa}{\phi_1}\sigma E^2 - \sup_E\left\{\frac{1}{2}\sigma E^2 - w_1(E)\right\}\right], \tag{55}$$

where $w_1(E)$ is the energy of the nonlinear conductor without cracks. Following Refs. 21 and 22, one can evaluate expression (55) as

$$\hat{W}(\overline{\mathbf{E}}) \geqslant \phi_1 w_1(\sqrt{A_\kappa/\phi_1}E), \quad (d=3). \tag{56}$$

Similar bounds for the two-dimensional nonlinear conductor with voids or cracks can be easily obtained by using the same procedure:

$$\hat{W}(\overline{\mathbf{E}}) \geqslant \phi_1 w_1(\sqrt{A_k/\phi_1}E), \quad (d=2), \tag{57}$$

$$\hat{W}(\overline{\mathbf{E}}) \geqslant \phi_1 w_1(\sqrt{A_\mu/\phi_1}E), \quad (d=2), \tag{58}$$

$$\hat{W}(\overline{\mathbf{E}}) \geqslant \phi_1 w_1(\sqrt{A_E/\phi_1}E), \quad (d=2). \tag{59}$$

The case of cracks in the material corresponds to the limit $\phi_1=1$. We emphasize that inequalities (56)–(59) are, to our knowledge, the first rigorous lower bounds on the effective energy of a nonlinear conductor with voids or cracks.

Figure 7 illustrates our inequalities for a composite made of a matrix with a power-law energy (46) ($n=3$) weakened by an equilibrium array of nonoverlapping spherical pores. The dashed curve shows the approximation[1] for the scaled conductivity constant σ_e/σ_1 based on the linear comparison material properties given by the Torquato[31] formula. The solid curve is the corresponding cross-property bound (56) with the bulk modulus given by the Torquato[28] approximation for the same composite made of a solid phase with the Poisson's ratio $\nu_1=0.3$. One can see that the bound and the approximation agree.

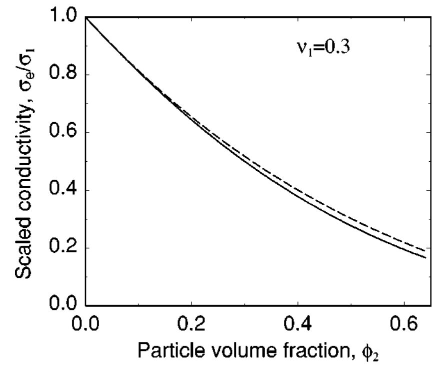

FIG. 7. Scaled conductivity coefficient σ_e/σ_1 vs the particle volume fraction ϕ_2 for random equilibrium arrays of spherical voids in a matrix with a power-law energy ($n=3$). The dashed curve is the approximation (Ref. 1) for the nonlinear energy of the porous conductor based on the Torquato (Ref. 30) approximation for effective conductivity. The solid curve is the cross-property bound (56) based on the Torquato (Ref. 28) approximation for the effective bulk modulus of this composite.

Figure 8 illustrates our results for a composite made of a matrix with a power-law energy (46) ($n=3$) weakened by randomly distributed penny-shaped noninteracting cracks. For noninteracting cracks, $\phi_1=1$, and the effective conductivity and bulk modulus of the corresponding linear material is given by

$$\frac{1}{\sigma_e}-\frac{1}{\sigma_1}=\frac{8}{9\sigma_1}\rho, \quad \frac{1}{\kappa_e}-\frac{1}{\kappa_1}=\frac{4(3\kappa_1+4\mu_1)}{3\mu_1(3\kappa_1+\mu_1)}\rho, \tag{60}$$

where ρ is the crack density.[17,32] The solid curve in Fig. 8 shows our lower bound (56) for the scaled effective conductivity constant σ_e/σ_1 in terms of the scaled effective bulk modulus κ_e/κ_1 of the same composite. The dashed curve is the corresponding cross-property approximation

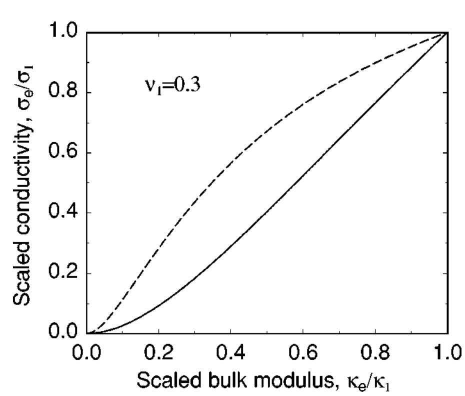

FIG. 8. Scaled conductivity coefficient σ_e/σ_1 vs the scaled effective bulk modulus κ_e/κ_1 for a composite of a matrix with a power-law energy (46) ($n=3$) weakened by randomly distributed penny-shaped noninteracting cracks. The solid curve is our lower bound (56), the dashed curve is the corresponding cross-property approximation (61).

$$\hat{W}(\overline{\boldsymbol{E}}) \geqslant w_1(\sqrt{A_N} E),$$
$$A_N = \frac{3\kappa_1\kappa_e(3\kappa_1+4\mu_1)}{3\kappa_1^2(2\mu_1+3\kappa_e)+2\kappa_1\mu_1(\mu_1+3\kappa_e)-2\mu_1^2\kappa_e} \quad (d=3), \tag{61}$$

which can be derived in a similar manner to the bound (56) by using the relations (60) for the effective energy of the linear comparison material, and excluding crack density ρ in favor of the effective bulk modulus. We assume that the composite is made of a solid phase with the Poisson's ratio $\nu_1 = 0.3$.

Note that the inequalities (55)–(59) applied to a linear conductor with the quadratic energy recover the bounds (51)–(54), respectively. The general case of two-phase composites with nonlinear energy is more difficult to treat, and we will not attempt to do it here.

IV. CONCLUSIONS

In this article we developed approximations on the effective complementary energy of incompressible two-phase d-dimensional elastic materials. For power-law materials, the scaled shear compliance μ_1/μ_e increases with the exponent n if the dispersed particles are more compliant than the matrix, and decreases with n if the dispersed particles are stiffer than the matrix. For finite ratio of the phase properties, the scaled effective shear compliance μ_1/μ_e very weakly depends on the exponent n if $n \geqslant 3$, as can be seen in Fig. 3. For the cases considered, our approximation lies within the best available rigorous bounds on the effective energy. We also developed a cross-property lower bound for the effective conductivity of a nonlinear conductor with voids and/or cracks. This is equal to the energy of an uncracked conductor with an argument (magnitude of the average electric field) normalized by a coefficient that depends on the effective bulk or shear modulus of the same composite. To our knowledge, this is the first rigorous lower bound on the effective energy of porous conductors with cracks.

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