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EFFECTIVE MECHANICAL AND TRANSPORT PROPERTIES OF CELLULAR SOLIDS

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Abstract—We utilize two different approaches, homogenization theory and discrete network analyses, to study the mechanical and transport properties of two-dimensional cellular solids (honeycombs) consisting of either hexagonal, triangular, square or Voronoi cells. We exploit results from homogenization theory for porous solids (in the low-density limit) to establish rigorous bounds on the effective thermal conductivity of honeycombs in terms of the elastic moduli and *vice versa*. It is shown that for hexagonal, triangular or square honeycombs, the cross-property bound relating the bulk modulus to the thermal conductivity turns out to be an exact and optimal result. The same is true for the cross-property bound linking the shear or Young's modulus of the triangular honeycomb to its conductivity. For low-density honeycombs, we observe that all of the elastic moduli do not depend on the Poisson's ratio of the solid phase. The elastic–viscoelastic correspondence principle enables us to conclude that all of the viscoelastic moduli of honeycombs in the low-density limit are proportional to the complex Young's modulus of the solid phase. Such structures have real Poisson's ratios and the loss tangent is the same for any load. © 1997 Published by Elsevier Science Ltd.

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1. INTRODUCTION

Cellular solids abound in nature and are fabricated by man on a large scale. Examples include wood, cancellous bone, cork, foams for insulation and packaging, sandwich panels in aircraft, and filters, to mention but a few [1].

We consider two-dimensional cellular solids (honeycombs) with hexagonal, triangular, square, or Voronoi cells in which the cell wall (solid) material is isotropic. The solid phase is characterized by its conductivity σ, bulk modulus κ, and shear modulus μ. An elastic isotropic material can also be characterized by its Young's modulus E and Poisson's ratio v. These parameters are related to the bulk and shear moduli via the expressions

$$E = \frac{4\kappa\mu}{\kappa + \mu}, \quad v = \frac{\kappa - \mu}{\kappa + \mu}. \tag{1}$$

Generally, cellular solids have small relative densities or, equivalently, small solid volume fractions.

In this paper, we consider two complementary approaches used to study the mechanics of cellular solids: *homogenization theory and discrete network analyses*. Under the assumption that the length scale associated with heterogeneities is much smaller than the macroscopic size of the composite, homogenization theory enables one to treat the composite as an effective homogeneous medium with a set of effective properties and to estimate the effective properties rigorously. The cellular solid from this point of view is a special two-phase composite, namely, a porous solid with a high porosity. In the discrete approach, the cellular solid is regarded to be a network of beams or channels. The effective behavior of the network is governed by the relevant physical mechanisms (such as deformation or transport) that occur in individual beams or channels.

Our main results concern the formulation of cross-property bounds that link the effective elastic moduli of honeycombs to the effective thermal conductivity (and *vice versa*) and estimates of the viscoelastic properties of honeycombs. The cross-property bounds are obtained by considering relations developed for general two-phase, isotropic composites [2–4]. The accuracy of the cross-property bounds is assessed using results form discrete network analyses. Accurate cross-property

relations are very useful if one property is easier to measure than the other or when only one of the properties is known. We also apply homogenization theory, discrete network analyses and the elastic–viscoelastic correspondence principle to find the viscoelastic properties of honeycombs.

The outline of the remainder of the paper is as follows: In Section 2 we discuss homogenization theory results that are especially useful for application to cellular solids. We study the elastic moduli and thermal conductivity of hexagonal, triangular, square, and Voronoi honeycombs in Sections 3–6, respectively. In Section 7, we state new conjectures regarding the elastic and thermal properties of honeycombs as well as new conjectures and principles regarding the viscoelastic properties of honeycombs.

2. HOMOGENIZATION THEORY RESULTS FOR HONEYCOMBS

The structure of cellular solids ranges from those characterized by a high degree of order to a high degree of disorder. We consider models of two-dimensional, honeycomb-like cellular solids comprised of periodic hexagonal, triangular, square, or Voronoi cells (Fig. 1) with a cell wall thickness of t and an edge of length l.

Since many real honeycombs have very small solid volume fractions, it is common to consider cases when the wall thickness t is much smaller than an edge of length l, i.e. $t/l \ll 1$. It is known that for such networks, simple beam theory gives results which are in good agreement with experiments [1]. However, homogenization theory offers rigorous results that are helpful as a benchmark for testing approximations or as a guideline for other approaches.

In this section we give a summary of some well-known and recent rigorous results from homogenization theory applied to the case of honeycombs, i.e. for two-dimensional porous composites in the limit that the solid volume fraction ϕ is very small. We shall henceforth refer to this limit as the *low-density asymptotic limit*, i.e. the leading term in the asymptotic expansion.

2.1. Hashin–Shtrikman bounds

The most well-known bounds in the theory of composites are the Hashin–Shtrikman [5–7] bounds on the effective properties of isotropic two-phase composites. In the limit when the properties of one of the phases (voids) are equal to zero, the upper bounds read as follows:

$$\frac{\sigma_*}{\sigma} \leqslant \frac{\phi}{2 - \phi}, \tag{2}$$

$$\frac{\kappa_*}{\kappa} \leqslant \frac{\phi\mu}{(1 - \phi)\kappa + \mu}, \tag{3}$$

$$\frac{\mu_*}{\mu} \leqslant \frac{\phi\kappa}{(1 - \phi)(\kappa + 2\mu) + \kappa}, \tag{4}$$

$$\frac{E_*}{E} \leqslant \frac{\phi}{3 - 2\phi}, \tag{5}$$

where σ_*, κ_*, μ_*, and E_* are the conductivity, bulk, shear, and Young's moduli of the composite, and ϕ is the volume fraction of the solid phase. The corresponding lower bounds are trivially equal to zero.

In the low-density asymptotic limit $\phi \ll 1$, the bounds (2)–(5) become

$$\frac{\sigma_*}{\sigma} \leqslant \frac{\phi}{2}, \tag{6}$$

$$\frac{\kappa_*}{\kappa} \leqslant \frac{\mu}{\kappa + \mu}\phi \quad \left[\text{or } \frac{\kappa_*}{E} \leqslant \frac{\phi}{4}\right], \tag{7}$$

$$\frac{\mu_*}{\mu} \leqslant \frac{\kappa}{2(\kappa + \mu)}\phi \quad \left[\text{or } \frac{\mu_*}{E} \leqslant \frac{\phi}{8}\right], \tag{8}$$

$$\frac{E_*}{E} \leqslant \frac{\phi}{3}. \tag{9}$$

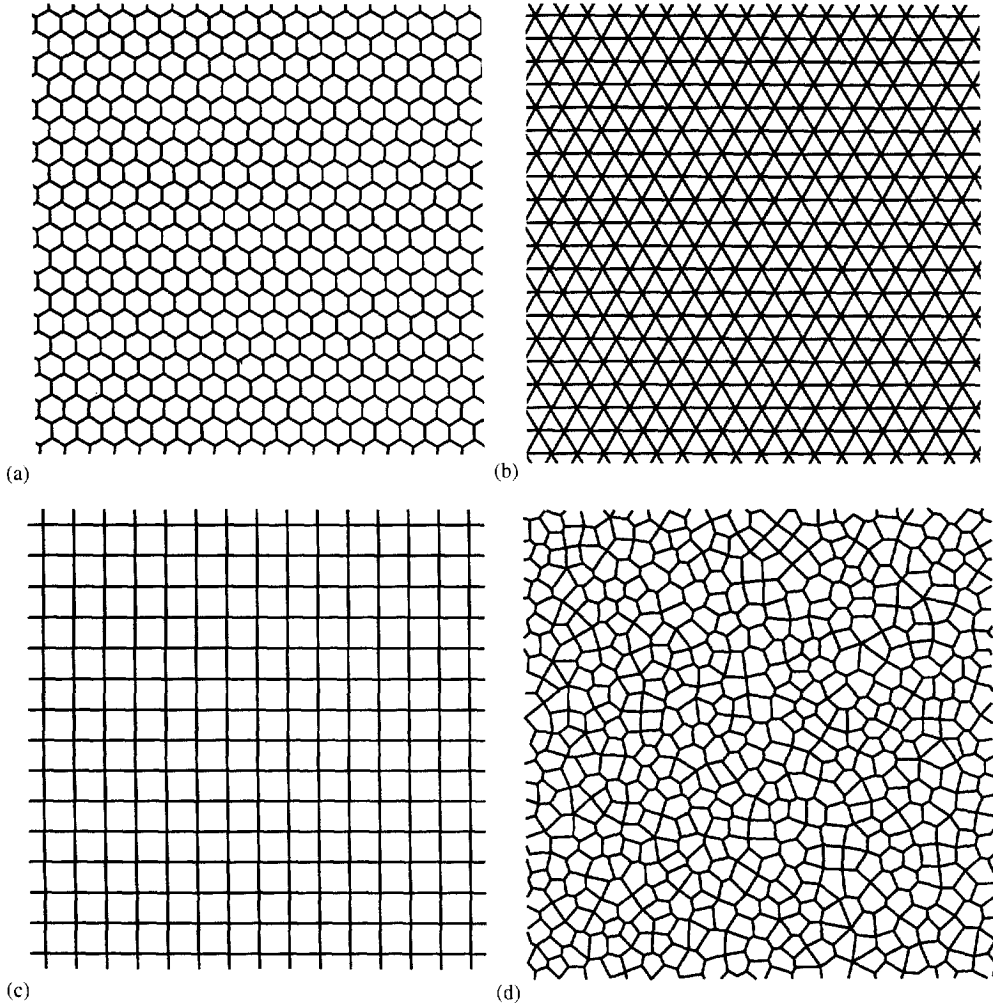The expressions within the brackets are useful alternative forms of the bounds.

Fig. 1. Cellular solids having (a) regular hexagonal cells, (b) equilateral triangular cells, (c) square cells, and (d) Voronoi cells.

The effective moduli of cellular solids obviously should satisfy the Hashin–Shtrikman bounds and thus the bounds may be used to test network theory results.

2.2. Cross-property conductivity-elastic moduli bounds

For composite media consisting of pores or cracks of arbitrary shape and size distributed throughout a solid material, Gibiansky and Torquato [4] derived cross-property bounds that relate the effective elastic moduli to the effective conductivity. The general cross-property bounds involving the effective bulk modulus, shear modulus, or Young's modulus read

$$\frac{\kappa}{\kappa_*} - 1 \geqslant \frac{\kappa + \mu}{2\mu}\left[\frac{\sigma}{\sigma_*} - 1\right], \tag{10}$$

$$\frac{\mu}{\mu_*} - 1 \geqslant \frac{\kappa + \mu}{\kappa}\left[\frac{\sigma}{\sigma_*} - 1\right], \tag{11}$$

$$\frac{E}{E_*} - 1 \geqslant \frac{3}{2}\left[\frac{\sigma}{\sigma_*} - 1\right]. \tag{12}$$

In the low-density asymptotic limit, i.e. $\phi \ll 1$, one can assume that $\sigma_*/\sigma \ll 1$, $\kappa_*/\kappa \ll 1$, $\mu_*/\mu \ll 1$, and $E_*/E \ll 1$. Under such conditions, the cross-property bounds (10)–(12) reduce to

$$\frac{\kappa_*}{\kappa} \leqslant \frac{2\mu}{\kappa + \mu} \frac{\sigma_*}{\sigma} \quad \left[\text{or } \frac{\kappa_*}{E} \leqslant \frac{1}{2} \frac{\sigma_*}{\sigma} \right], \tag{13}$$

$$\frac{\mu_*}{\mu} \leqslant \frac{\kappa}{\kappa + \mu} \frac{\sigma_*}{\sigma} \quad \left[\text{or } \frac{\mu_*}{E} \leqslant \frac{1}{4} \frac{\sigma_*}{\sigma} \right] \tag{14}$$

and

$$\frac{E_*}{E} \leqslant \frac{2}{3} \frac{\sigma_*}{\sigma}, \tag{15}$$

respectively. Note that measurement of the elastic moduli in conjunction with the cross-property bounds (10)–(15) allows one to obtain a *lower bound* on the effective conductivity. Similarly, conductivity information and bounds (10)–(15) enable one to bound the elastic moduli from above.

It is noteworthy that the general bounds (7)–(9) and bounds (13)–(15) can be written in a form that does not include the Poisson's ratio of the solid phase. We will return to this interesting point in the next section. One should also note that the bounds on the effective Young's modulus (5) and (12) also do not depend on the Poisson's ratio of the solid phase. This is a consequence of a more general principle that we state below.

2.3. Cherkaev–Lurie–Milton (CLM) principle

We have seen that the Hashin–Shtrikman bound (5) for the effective Young's modulus, as well as the cross-property bound (12) do not depend on the Poisson's ratio of the solid phase. A similar observation was made by Day *et al.* [8], who found numerically that the effective Young's modulus of a two-dimensional solid with holes does not depend on the Poisson's ratio of the solid phase. This numerical result was confirmed theoretically by Cherkaev *et al.* [9], who based their study on the earlier work by Lurie and Cherkaev [10]. The CLM principle (which is not stated here explicitly) implies that *the effective Young's modulus of isotropic porous materials (including honeycombs) at arbitrary volume fraction ϕ does not depend on the Poisson's ratio of the solid phase*. The CLM principle was used to study the effective properties of porous two-dimensional materials by Thorpe and Jasiuk [11]. Christensen [12] applied it to transversely isotropic porous materials with cylindrical pores and observed that it is approximately valid for three-dimensional porous composites with positive Poisson's ratio of the solid phase. The CLM principle can be used to check theoretical or numerical results.

2.4. Vigdergauz microstructures

Recently, Vigdergauz found the optimal shape of a hole in a square array of such holes [13, 14] and the optimal shape of a hole in a hexagonal array of such holes [15] such that the effective bulk modulus satisfies the Hashin–Shtrikman bound (3) exactly. It is intuitively clear that in the low-density asymptotic limit, the effective bulk moduli of these composites should asymptotically approach those of square and hexagonal honeycombs, respectively. Therefore, square or hexagonal honeycombs should have effective bulk moduli that are very close to the Hashin–Shtrikman bulk modulus bound. Indeed, we will see that they satisfy the bound exactly.

2.5. Elastic–viscoelastic correspondence principle

We will also consider estimating the viscoelastic properties of honeycombs. To do so we make use of the correspondence principle between the elastic and viscoelastic behavior of composites: the effective complex viscoelastic moduli of a composite (or p times Laplace transforms of the effective viscoelastic moduli, where p is a transform variable) can be found from formulas for the effective elastic moduli by replacing the real moduli in these expressions by their complex counterparts (or by p times Laplace transforms of the viscoelastic moduli of phases), see, e.g., Refs. [16] or [17]. Thus, all of the analytical expressions for the effective elastic moduli can be used to obtain corresponding effective viscoelastic moduli.

In what follows, we will apply all of the aforementioned results to study the effective mechanical and transport properties of hexagonal, triangular, square, and Voronoi honeycombs.

3. HEXAGONAL HONEYCOMBS

Consider a regular hexagonal honeycomb shown in Fig. 1(a). The relative density of the cellular solid (defined to be the ratio of the composite density to the solid density) is exactly equal to the solid volume fraction ϕ which is, for the hexagonal honeycomb in the low-density asymptotic limit $\phi \ll 1$, given by

$$\phi = \frac{2}{\sqrt{3}}\frac{t}{l}. \tag{16}$$

Such honeycombs obviously have hexagonal symmetry and thus are macroscopically isotropic with respect to conduction and elastic properties. It is straightforward to obtain analytically expressions for the in-plane effective conductivity and in-plane effective elastic moduli. For hexagonal honeycombs in the low-density asymptotic limit, the effective conductivity σ_* is given by

$$\frac{\sigma_*}{\sigma} = \frac{1}{\sqrt{3}}\frac{t}{l} = \frac{\phi}{2} \tag{17}$$

and the effective bulk modulus κ_* is given by [1]

$$\frac{\kappa_*}{E} = \frac{1}{2\sqrt{3}}\frac{t}{l} = \frac{\phi}{2}\left[\text{ or } \frac{\kappa_*}{\kappa} = \frac{\mu}{\kappa + \mu}\phi\right]. \tag{18}$$

The effective shear modulus μ_* and the Young's modulus E_* of a regular hexagonal honeycomb in the low-density asymptotic limit $\phi \ll 1$ are, respectively, given by [1]

$$\frac{\mu_*}{E} = \frac{1}{\sqrt{3}}\left(\frac{t}{l}\right)^3 = \frac{3}{8}\phi^3, \tag{19}$$

$$\frac{E_*}{E} = \frac{4}{\sqrt{3}}\left(\frac{t}{l}\right)^3 = \frac{3}{2}\phi^3. \tag{20}$$

It is of interest to compare these expressions with the appropriate limits (6)–(9) of the Hashin–Shtrikman upper bounds. (Recall that the lower bounds are trivially equal to zero.) Comparing relations (6)–(9) with (17)–(20) reveals that the hexagonal honeycomb has *extermal conductivity and bulk modulus values*, i.e. the expressions coincide with the appropriate bounds. However, the shear modulus and the Young's modulus of the honeycomb are much smaller than the theoretical upper bound. The reason is clear: the hexagonal structure is much more compliant under shear loading than under hydrostatic loading due to bending of the cell walls under shear. In the next section, we will consider triangular honeycombs that turn out to have optimal values of the effective conductivity as well as all of the effective elastic moduli.

Note that the cross-property bound (13) coincides with the exact relation

$$\frac{\kappa_*}{E} = \frac{1}{2}\frac{\sigma_*}{\sigma}, \tag{21}$$

that follows from (17) and (18) (see Fig. 2). We will now combine these known results on the effective elastic moduli of hexagonal honeycombs and cross-property bounds (10)–(15) to predict the effective conductivity of such cellular solids, assuming that we did not have the knowledge of the exact result (17).

One can substitute expression (18) for the effective bulk modulus of a hexagonal honeycomb into relation (13) to get the *lower bound* on the effective conductivity of the hexagonal honeycombs

$$\frac{\sigma_*}{\sigma} \geqslant \frac{\kappa + \mu}{2\mu}\frac{\kappa_*}{\kappa} = \frac{\phi}{2}, \tag{22}$$

see Fig. 2. Using this cross-property lower bound in conjunction with the Hashin–Shtrikman *upper bound* on the effective conductivity given by Eqn (6) yields

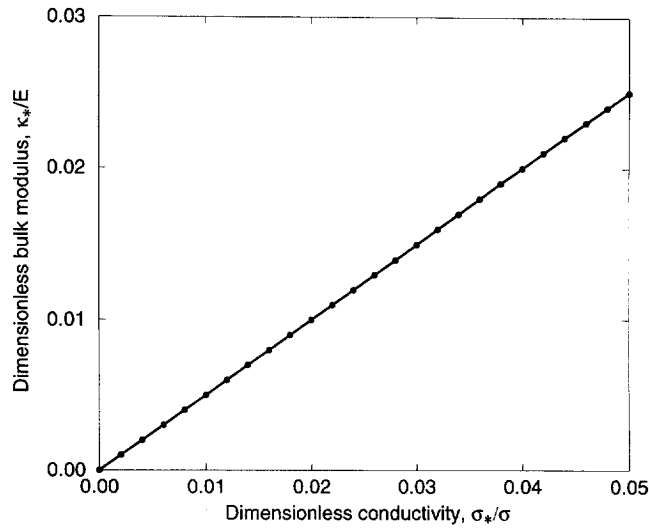$$\frac{\sigma_*}{\sigma} = \frac{\phi}{2}, \tag{23}$$

which we see is exact!

Fig. 2. Dimensionless bulk modulus versus dimensionless conductivity for hexagonal honeycombs. Circles are exact results taken from Eqn (21) and the solid line is the cross-property upper bound (13).

Why is the cross-property bound (13) involving κ_* and σ_* exact? As noted by Gibiansky and Torquato [4], the general bound (10) is exact for the structures that satisfy Hashin–Shtrikman conductivity and bulk modulus bounds (6), (7). Therefore, they are exact for a whole range of structures, including the Hashin [7] space-filling assemblages of coated circles when the void phase forms the inner circle, Vigdergauz structures with square cell [13, 14] or hexagonal cell [15], and certain porous laminate composites [18]. As we have mentioned, the conductivity and bulk modulus of hexagonal honeycombs are equal to the Hashin–Shtrikman bounds in the low-density asymptotic limit ($\phi \ll 1$) and thus satisfy the bound (13) exactly.

Now let us consider obtaining the exact relation connecting the effective Young's modulus E_* to the effective conductivity σ_* of a hexagonal honeycomb. The effective Young's modulus for a hexagonal honeycomb is given by Eqn (20). Use of expression (17) yields the exact cross-property relation linking the effective Young's modulus to the effective conductivity as

$$\frac{E_*}{E} = 12\left(\frac{\sigma_*}{\sigma}\right)^3 . \tag{24}$$

The cross-property inequality (15) bounds the exact relation (24) for the effective conductivity from above. However, referring to Fig. 3, it is clear that this cross-property bound provides a poor estimate of the effective property.

Why is the cross-property bound (15) involving E_* and σ_* a poor estimator? Gibiansky and Torquato [4] observed that the general bound (12) is exact for certain porous laminate composites [18] which are structurally very different from hexagonal honeycombs. We will examine triangular honeycomb structures that are similar to such porous laminate composites in the next section.

It is interesting to observe that the expression (20) for the effective Young's modulus of the regular honeycomb structures in the low-density asymptotic limit satisfies the CLM principle. Moreover, note that all of the elastic moduli of the regular honeycombs are independent of the Poisson's ratio of the solid phase. The reason for this is that the stiffness of cellular solids is governed by extension or bending of the cell walls which is purely a function of E and hence only the Young's modulus of the solid phase should appear in the final formulas. This is not true for the effective bulk modulus κ_* and shear modulus μ_* for *arbitrary* ϕ. In fact, for general ϕ, the Hashin–Shtrikman bounds (3)–(4) on κ_* and μ_* as well as cross-property bounds (10)–(11) depend not only on the Young's modulus E but also on the Poisson's ratio ν of the solid. However, it follows from the CLM principle that the effective Young's modulus E_* of a honeycomb at arbitrary ϕ should not depend on ν.
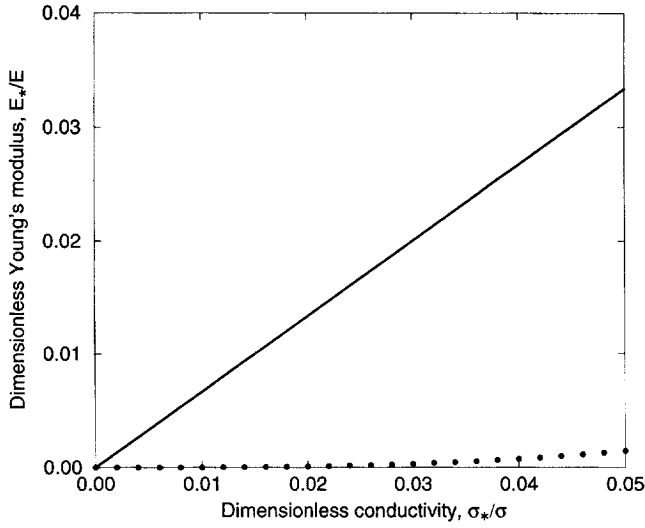
Fig. 3. Dimensionless Young's modulus versus dimensionless conductivity for hexagonal honeycombs. Circles are exact results taken from Eqn (24) and the solid line is the cross-property upper bound (15).

In summary, the effective bulk modulus and effective conductivity of periodic honeycombs correlate exactly. Given a measurement of the effective bulk modulus, the effective conductivity can be predicted exactly. Given a measurement of the effective conductivity, one can obtain the upper bound on the effective bulk modulus. This bound is equivalent to the Hashin–Shtrikman bulk modulus bound (7). Thus, the conductivity measurement is equivalent to the volume fraction measurement in this instance.

By contrast, the corresponding effective Young's modulus of periodic honeycombs does not correlate well with the effective conductivity.

4. TRIANGULAR HONEYCOMBS

Consider now periodic honeycombs with equilateral triangular cells [see Fig. 1(b)]: the dual lattice to the aforementioned hexagonal structure. The solid volume fraction of such cellular solids in the low-density asymptotic limit $\phi \ll 1$ is given by

$$\phi = 2\sqrt{3}\frac{t}{l}. \tag{25}$$

Such a lattice has hexagonal symmetry and thus has isotropic conductivity and isotropic elastic properties. In the low-density asymptotic limit, the effective bulk, shear, and Young's modulus of triangular honeycombs are given by [19]

$$\frac{\kappa_*}{E} = \frac{\sqrt{3}}{2}\frac{t}{l} = \frac{\phi}{4} \quad \left[\text{or } \frac{\kappa_*}{\kappa} = \frac{\mu}{\kappa + \mu}\phi \right], \tag{26}$$

$$\frac{\mu_*}{E} = \frac{\sqrt{3}}{4}\frac{t}{l} = \frac{\phi}{8} \quad \left[\text{or } \frac{\mu_*}{\mu} = \frac{\kappa}{2(\kappa + \mu)}\phi \right] \tag{27}$$

and

$$\frac{E_*}{E} = \frac{2}{\sqrt{3}}\frac{t}{l} = \frac{\phi}{3}, \tag{28}$$

respectively. Note that Eqn (28) corrects the result given by Gibson and Ashby [1].

Comparing the relations above with the Hashin–Shtrikman bounds (7)–(9), shows that the effective bulk, shear, and Young's moduli of the triangular-honeycomb have extremal values. The

reason is clear: triangular honeycombs are stiff under both bulk and shear deformations. Indeed, civil engineering structures, such as bridges and towers, are often based on triangular cell structures. The fact that triangular honeycombs have elastic properties that achieve the Hashin–Shtrikman bounds was also noted by Christensen [12].

Let us now predict the effective conductivity of triangular honeycombs using elasticity information and cross-property bounds. Substitution of the elastic moduli relations (26)–(28) into the corresponding cross-property bounds (13)–(15) give the same *lower bound*, i.e.

$$\frac{\sigma_*}{\sigma} \geqslant \frac{\kappa + \mu}{2\mu} \frac{\kappa_*}{\kappa} = \frac{\phi}{2}, \tag{29}$$

$$\frac{\sigma_*}{\sigma} \geqslant \frac{\kappa + \mu}{\kappa} \frac{\mu_*}{\mu} = \frac{\phi}{2}, \tag{30}$$

$$\frac{\sigma_*}{\sigma} \geqslant \frac{3}{2} \frac{E_*}{E} = \frac{\phi}{2}. \tag{31}$$

This lower bound together with the Hashin–Shtrikman *upper bound* (6) yields the following exact result on the effective conductivity:

$$\frac{\sigma_*}{\sigma} = \frac{\phi}{2} = \sqrt{3} \frac{t}{l}. \tag{32}$$

Thus, the effective bulk modulus, shear modulus, or Young's modulus cross-property bounds in conjunction with the Hashin–Shtrikman conductivity bound allow us to predict the effective conductivity of triangular honeycombs exactly.

Measurement of the effective conductivity leads to upper bounds on the effective elastic moduli that coincide with the appropriate limits of the Hashin–Shtrikman bounds on these moduli. Thus, such a conductivity measurement is equivalent to a volume fraction measurement for triangular honeycombs. Note that the elastic moduli of the triangular honeycomb do not depend on the Poisson's ratio v for the reasons already discussed in the previous section.

The Vigdergauz-type structures are not known for composites with a triangular cell. If they exist, they should be geometrically similar to the triangular honeycombs in the low-density limit. On the other hand, in the opposite limit of high density, i.e. $1 - \phi \ll 1$, such triangular cell Vigdergauz composites would be equivalent to a composite with hexagonal array of small circular holes. Thus, in both limits such composites would not only have extremal bulk modulus and conductivity, but also extremal Young's and shear moduli. *If such composites exist, they may have extremal conductivity, bulk, shear, and Young's moduli for a wide range of volume fractions.*

5. SQUARE HONEYCOMBS

In this section we consider periodic honeycombs with square cells [see Fig. 1(c)]. The volume fraction of the solid phase in the low-density asymptotic limit $\phi \ll 1$ is given by

$$\phi = \frac{2t}{l}. \tag{33}$$

Such structures possess only square symmetry, and thus the effective conductivity tensor is isotropic but the effective elastic tensor possesses only square symmetry. The elastic properties of square honeycombs are characterized by the effective bulk modulus κ_*, and two different effective shear moduli $\mu_*^{(1)}, \mu_*^{(2)}$. The modulus $\mu_*^{(1)}$ characterizes the response of the square honeycomb when it is sheared along the sides of the cell, whereas $\mu_*^{(2)}$ characterizes the response when it is sheared along the diagonals of the square cell. Note that the Hashin–Shtrikman shear or Young's modulus bounds (4)–(5) cannot be applied to square honeycombs since such structures are not elastically isotropic. In the low-density asymptotic limit, the effective bulk modulus is given by [1]

$$\frac{\kappa_*}{E} = \frac{\phi}{4} = \frac{t}{2l}. \tag{34}$$

We recall the earlier observation that Vigdergauz constructions with square cells realize the Hashin–Shtrikman bounds for any volume fraction and thus in the low-density asymptotic limit must equal Eqn (7). As we expected, the bulk modulus of the Vigdergauz structure is equal to the bulk modulus of the square honeycomb (34).

Noting that the cross-property bound (13) involving κ_* and σ_* is also valid for square symmetric composites, we apply this bound to obtain the conductivity of square honeycombs to be given by

$$\frac{\sigma_*}{\sigma} = \frac{\phi}{2}, \tag{35}$$

as was done for the hexagonal and triangular honeycombs.

Similar reasoning enables us to deduce the exact result for the effective conductivity of a Vigdergauz construction at any volume fraction ϕ. Specifically, it is given by the Hashin–Shtrikman expression (2), i.e.

$$\frac{\sigma_*}{\sigma} = \frac{\phi}{2 - \phi}. \tag{36}$$

6. RANDOM HONEYCOMBS

Silva et al. [20] numerically generated random, two-dimensional, isotropic honeycombs using Voronoi cells with uniform thickness and performed finite-element analyses to determine the in-plane effective elastic moduli. The "nucleation" points were generated randomly and sequentially such that each point was accepted if it was outside a minimum distance from other point [see Fig. 1(d)]. This is a special Voronoi tessellation corresponding to a random sequential addition of hard disks. The Young's modulus of the cell wall material was taken to be 1, whereas the corresponding Poisson's ratio was taken to be 0.3. They studied 20 different isotropic realizations of this random honeycomb structure with a solid volume fraction $\phi = 0.15$. The average values of the effective elastic constants were found to be

$$\frac{E_*}{E} = 0.00489, \tag{37}$$

$$v_* = 0.926. \tag{38}$$

From the interrelations

$$\kappa_* = \frac{E_*}{2(1 - v_*)}, \quad \mu_* = \frac{E_*}{2(1 + v_*)}, \tag{39}$$

we find that the effective bulk and shear moduli are given by

$$\frac{\kappa_*}{E} = 0.03304, \quad \frac{\mu_*}{E} = 0.00127. \tag{40}$$

Note that the bulk modulus value is below but not far from the Hashin–Shtrikman upper bound κ_*/E of 0.04155, whereas the shear modulus is much smaller than the Hashin–Shtrikman upper bound μ_*/E of 0.02086, as expected.

Silva et al. did not compute the effective conductivity of this random honeycomb. Given our conclusions about periodic honeycombs, we expect that the effective bulk modulus value (40) in conjunction with the cross-property relation (10) will yield a sharp estimate of the effective conductivity σ_*. We can recast cross-property bound (10) as

$$\frac{E}{\kappa_*} - \frac{E}{\kappa} \geqslant 2\left[\frac{\sigma}{\sigma_*} - 1\right]. \tag{41}$$

Using the fact that $E/\kappa = 2(1 - v) = 1.4$ for this composite, we determine that the effective conductivity is bounded from below according to the relation

$$\frac{\sigma_*}{\sigma} \geqslant 0.0648. \tag{42}$$

The conductivity-shear modulus bound (11) with the effective shear modulus given by Eqn (40) leads to the conductivity bound

$$\frac{\sigma_*}{\sigma} \geqslant 0.0051, \tag{43}$$

which is a poor estimate compared to Eqn (42).

The Hashin–Shtrikman upper bound (2) on the effective conductivity for such a composite reads as follows:

$$\frac{\sigma_*}{\sigma} \leqslant \frac{\phi}{2 - \phi} = 0.0811. \tag{44}$$

Thus, we see that the lower bound (42) on σ_* is sharp.

7. DISCUSSION

7.1. *Thermal and elastic properties of honeycombs*
We have applied homogenization theory and discrete network analyses to study the elastic and thermal properties of cellular solids. It has been shown that the effective conductivity and effective bulk modulus of regular honeycombs with a hexagonal, triangular, or square cell in the low-density limit ($\phi \ll 1$) have extremal values that coincide with the Hashin–Shtrikman upper bounds on the conductivity and bulk modulus, respectively. Moreover, the effective shear and Young's moduli of triangular honeycombs have extremal values that coincide with the Hashin–Shtrikman upper bounds on these moduli. In these situations, cross-property bounds linking the elastic properties and the conductivity are exact.

However, cross-property bounds provide a poor correlation between the shear modulus and conductivity of hexagonal or Voronoi honeycombs. The reason for this is that the shear modulus of such structures is governed by *bending resistance of the cell walls*, which is very different from axial transport that determines the conductivity.

We have observed that the expression for the effective Young's modulus of the honeycomb structures does not depend on the Poisson's ratio of the solid phase, as it should be according to the CLM principle. We also have noted that the asymptotic expressions to leading order in ϕ for the bulk and shear moduli of regular honeycombs (hexagonal, triangular, square) are also independent of the solid-phase Poisson's ratio v. This is so because both the tensile and bending resistance of the cell walls are determined by the solid-phase Young's modulus E and are independent of the Poisson's ratio v of the solid phase. *We conjecture that all of the effective moduli of any honeycomb structure (ordered or not) in the low-density asymptotic limit are independent of the Poisson's ratio v of the solid phase.* This is not true for κ_* and μ_* when ϕ is not small.

7.2. *Viscoelastic moduli of honeycombs*
It is useful to combine the results for the effective elastic moduli of honeycombs and the correspondence principle for viscoelasticity to obtain the associated viscoelastic moduli of the cellular solids. In particular, we immediately conclude that the effective bulk modulus of the hexagonal and square honeycombs and all of the elastic moduli of the triangular honeycomb, are given by the equalities in the Hashin–Shtrikman bounds (3)–(5) with complex values of the effective moduli and the complex Young's modulus of the solid phase. The other viscoelastic moduli of honeycombs can be obtained in a similar way from the corresponding expressions for the elastic moduli.

We have noted that the ratio E_*/E, *for porous composites at arbitrary volume fraction* ϕ, does not depend on the Poisson's ratio and therefore depends only on the microstructure of the composite. Out previous conjecture regarding the effective elastic moduli of honeycombs is equivalent to the statement that the ratios κ_*/E, μ_*/E, or even c^*_{ijkl}/E (where c^*_{ijkl} is any element of the effective elastic tensor) for honeycombs in the low-density asymptotic limit ($\phi \ll 1$) does not depend on the Poisson's ratio and therefore depends only on the honeycomb microstructure.

From the correspondence principle stated in Section 2, one can obtain the complex viscoelastic moduli (or p times Laplace transform of the viscoelastic moduli, where p is a transform variable) by simply substituting the complex Young's modulus $E = E' + iE''$ [or p times Laplace transform $\hat{E}(p)$

of the relaxation function $E(t)$] into the appropriate elastic formulas. Recall that the Poisson's ratio v does not enter these expressions.

Combining these observtion one may conclude the following.

(i) The ratio E_*/E for *two-dimensional viscoelastic porous composites with arbitrary density* is a real number that depends only on the composite microstructure. It is equal to the effective Young's modulus of the *elastic* composite of the same structure made of a solid phase with $E = 1$.

(ii) The ratios κ_*/E, μ_*/E, or even c_{ijkl}^*/E for *viscoelastic honeycombs in the low-density asymptotic limit* are real numbers that depend only on the composite microstructure and also can be found from the analyses of the elastic problem taking $E = 1$.

Here, κ_*, μ_*, E_*, or c_{ijkl}^* may be taken to be either the corresponding complex moduli or the Laplace transforms of the appropriate relaxation functions.

It follows from (ii) that *the Poisson's ratios of the low-density honeycombs are real. They depend only on the structure of the composite and independent of the viscoelastic properties of the solid phase.*

An interesting consequence of these statements is that the loss tangent of the low-density honeycombs for any mode of deformation is given by the loss tangent of the solid phase for uniaxial deformation, i.e.

$$\frac{\kappa_*'}{\kappa_*''} = \frac{\mu_*'}{\mu_*''} = \frac{E_*'}{E_*''} = \frac{E'}{E''}. \tag{45}$$

For the relaxation functions, the above statements lead to the following conclusions:

(i) For *two-dimensional viscoelastic porous composites with arbitrary density*, the ratio of the effective Young's relaxation modulus of the composite to that of the solid phase, $E_*(t)/E(t)$, is a real number that depends only on the composite microstructure. It is equal to the effective Young's modulus of the *elastic* composite of the same structure made of a solid phase with $E = 1$.

(ii) The ratios $\kappa_*(t)/E(t)$, $\mu_*(t)/E(t)$, or even $c_{ijkl}^*(t)/E(t)$ for *viscoelastic honeycombs in the low-density asymptotic limit* are real numbers that depend only on the composite microstructure and also can be found from the analyses of the elastic problem taking $E = 1$. Here $\kappa_*(t)$, $\mu_*(t)$, and $c_{ijkl}^*(t)$ are the corresponding effective relaxation functions.

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