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Optimal design of 1-3 composite piezoelectrics

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Abstract An optimal design problem for piezoelectric composite hydrophones is considered. The hydrophone consists of parallel piezoelectric rods embedded in a porous *transversely isotropic* polymer matrix. We find the shape, volume fraction, and spatial arrangement of the piezoceramic rods, and the structure of the matrix material that maximizes the hydrophone performance characteristics. We found that the optimal composite consists of a hexagonal array of rods with small volume fraction, in a highly anisotropic matrix that is characterized by negative Poisson's ratios in certain directions. The performance characteristics of hydrophones with such a matrix are significantly higher than those with an *isotropic* polymer matrix. The results can be viewed as theoretical upper bounds on the hydrophone performance.

1 Introduction

Piezoelectric transducers have been employed as sensors and transmitters of acoustic signals in ultrasound medical imaging, nondestructive testing and underwater acoustics (hydrophones). In this paper we consider the optimal design of a hydrophone composite consisting of parallel piezoceramic rods that are embedded in a porous polymer matrix. The hydrophone is assumed to operate in the low-frequency range and hence its behaviour can be described in the quasistatic limit.

One may ask why is pure piezoceramic not used since it is the only material with piezoelectric properties? The basic problem is that *under hydrostatic load*, the anisotropic piezoelectric response of pure piezoelectric is such that it has poor hydrophone performance characteristics such as hydrostatic piezoelectric coefficient d_h, *voltage coefficient* $g_h = d_h/\varepsilon_{33}$ (where ε_{33} is a dielectric constant in the x_3-direction), the *hydrophone figure* of *merit* $d_h g_h$, and the *electromechanical coupling factor* $k_h = \sqrt{d_h g_h / s_h}$ (where s_h is a dilatational compliance).

It was shown in a number of papers (see e.g. Klicker *et al.* 1981; Newnham 1986; Newnham and Ruschau 1991; Ting *et al.* 1990) that composites with high hydrophone sensitivity can be achieved by making a composite consisting of piezoceramic rods in a soft polymer matrix. Figure 1 schematically depicts such a "1-3 piezocomposite" when exposed to a hydrostatic pressure field. An appropriately designed piezocomposite is capable of converting an applied hydrostatic field into a predominantly tensile stress on the rods, thus enhacing all of the hydrophone characteristics. Using simple models in which the elastic and electric fields were taken to be uniform in the different phases, Haun and Newnham (1986), Chan and Unsworth (1989), and Smith (1991, 1993) qualitatively explained the enhancement due to the Poisson's ratio effect. Smith (1991) proposed that even greater enhancement in hydrophone characteristics can be achieved by using matrix materials with negative Poisson's ratio. A more sophisticated analysis was recently given by Avellaneda and Swart (1994) using the so-called *differential-effective-medium approximation.*

It was found that the performance of the composite depends significantly on the properties and the volume fraction of the rods, and on the mechanical properties of the polymer matrix. For example, the use of a matrix with negative Poisson's ratio or a porous matrix increases the sensitivity of the hydrophone by an order of magnitude.

This paper extends the analyses of Avellaneda and Swart (1994). Our main contribution is that we depart from the assumption of isotropy of the matrix, and require only *transverse isotropy* of this material. We treat the matrix material itself as a composite; it is assumed to be prepared from a polymer with given properties, weakened by an optimal arrangement of pores. The microstructure of the matrix material is an additional control in the problem that we study. As we will see, the optimal matrix is highly anisotropic, with a large ratio of the minimal and maximal eigenvalues of the stiffness tensor. Here we only give a summary of the results and a brief description of the method. The detailed derivation of our results will be published elsewhere (Gibiansky and Torquato 1997).

The paper is organized as follows. In Section 2, we give a brief summary of the formulae that describe performance characteristics of hydrophones. In Section 3, we discuss the design parameters of the problem. Section 4 presents the results of numerical optimization. Section 5 summarizes the results of the paper.

2 Hydrophone performance characteristics

In this section we give a brief summary of the formulae that describe piezoelectric hydrophones (see e.g. Smith 1991, 1993; Avellaneda and Swart 1994). The object under study is a composite of PZT-ceramic rods in a porous polymer. If the wavelength of the applied field is much larger than the spacing between rods, the behaviour of a composite can be characterized by the averaged equations of piezoelectricity, i.e.

$$\begin{pmatrix} \mathbf{S} \\ \mathbf{D} \end{pmatrix} = \begin{pmatrix} \mathbf{s}^E & \mathbf{d} \\ \mathbf{d}^t & \varepsilon^T \end{pmatrix} \begin{pmatrix} \mathbf{T} \\ \mathbf{E} \end{pmatrix}, \quad (1)$$

where $\mathbf{S}$ is the average strain tensor, $\mathbf{D}$ is the average dielectric diplacement vector, $\mathbf{T}$ is the average stress tensor, $\mathbf{E}$ is the average electrical field, $s^E = s^E_{ijk\ell}$ is a fourth-order effective compliance tensor under short circuit boundary conditions, $d = d6_{ijk}$ is a third-order effective piezoelectric stress coupling tensor and $\varepsilon^T = \varepsilon^T_{ij}$ is the second order free-body dielectric tensor. An alternative form of the same constitutive relations is

$$\begin{pmatrix} \mathbf{T} \\ \mathbf{D} \end{pmatrix} = \begin{pmatrix} \mathbf{c}^E & -\mathbf{e} \\ \mathbf{e}^t & \varepsilon^S \end{pmatrix} \begin{pmatrix} \mathbf{S} \\ \mathbf{E} \end{pmatrix}, \tag{2}$$

where $\mathbf{c}^E = \left(\mathbf{s}^E\right)^{-1}$ is the effective short-circuit stiffness tensor, $\varepsilon^S = \varepsilon^T - \mathbf{d}^t \left(\mathbf{s}^E\right)^{-1} \mathbf{d}$ is a clamped-body effective dielectric tensor, and $\mathbf{e} = \left(\mathbf{s}^E\right)^{-1} \mathbf{d}$ is the effective piezoelectric strain tensor. It is convenient to use dyadic notation for the problem under study, i.e.

$$S_i = (\mathbf{S})_{ii}, \quad e_{13} = (\mathbf{e})_{113}, \quad e_{33} = (\mathbf{e})_{333}, \quad s_{ij} = (\mathbf{s})_{iijj}, \quad i,j = 1,2,3, \tag{3}$$

etc., where the coefficients on the right-hand sides of (3) are the coefficients of the corresponding tensors in the Cartesian basis.

The response of the transversely isotropic hydrophone composite under hydrostatic pressure $\langle \mathbf{T} \rangle = T\delta_{ij}$ ($\delta_{ij} = 1$ if $i = j$ and $\delta_{ij} = 0$ otherwise) is commonly characterized by three quantities.

(i) *The hydrostatic coupling coefficient* defined by

$$d_h = \langle D_3 \rangle / T = 2d_{13} + d_{33} \tag{4}$$

measures the polarization sensitivity.

(ii) *The hydrophone figure of merit*

$$d_h g_h = d_h^2 / \varepsilon^T_{33} \tag{5}$$

is another measure of the sensitivity of the composite.

(iii) *The electromechanical coupling factor,* k_h, defined by

$$k_h = \sqrt{\frac{d_h^2}{\varepsilon^T_{33} s_h}}, \tag{6}$$

where k_h^2 measures the overall acoustic/electric power conversion. Here

$$s_h = 2s_{11} + 2s_{12} + 4s_{13} + s_{33} \tag{7}$$

is a dilatational compliance and s_{ij} are the dyadic coefficients of the tensor $\mathbf{s}^E$. These performance characteristics can be written in the following way

$$d_h = \boldsymbol{v} \cdot \mathbf{C}^{-1} \cdot \mathbf{e}, \quad d_h g_h = \frac{\left(\boldsymbol{v} \cdot \mathbf{C}^{-1} \cdot \mathbf{e}\right)^2}{\varepsilon_{33} + \mathbf{e} \cdot \mathbf{C}^{-1} \cdot \mathbf{e}},$$

$$k_h^2 = \frac{\left(\boldsymbol{v} \cdot \mathbf{C}^{-1} \cdot \mathbf{e}\right)^2}{\left(\varepsilon_{33} + \mathbf{e} \cdot \mathbf{C}^{-1} \cdot \mathbf{e}\right) \boldsymbol{v} \cdot \mathbf{C}^{-1} \cdot \boldsymbol{v}}, \tag{8}$$

where

$$\mathbf{C} = \begin{pmatrix} K & c_{13} \\ c_{13} & c_{33} \end{pmatrix}, \quad \mathbf{e} = (e_{13} \quad e_{33}), \quad \boldsymbol{v} = (1 \quad 1), \tag{9}$$

and $K = (c_{11} + c_{12})/2$ is the transverse bulk modulus. Note that the coefficients in the formulae (8) and (9) refer to the effective characteristics.

For transversely isotropic composites made of transversely isotropic PZT rods and a transversely isotropic polymer matrix, one can express the effective coefficients of the composite (those that are relevant for our analyses) in terms of the coefficients of the PZT-ceramics, polymer matrix, and the only structural parameter p:

$$\begin{aligned}
c_{13} &= c^m_{13} + fp\left(c^r_{13} - c^m_{13}\right), \\
e_{13} &= e^m_{13} + fp\left(e^r_{13} - e^m_{13}\right), \\
c_{33} &= c^m_{33} + f\left[c^r_{33} - c^m_{33} + (p-1)\frac{(c^r_{13} - c^m_{13})^2}{(K^r - K^m)}\right], \\
e_{33} &= e^m_{33} + f\left[e^r_{33} - e^m_{33} + (p-1)\frac{(c^r_{13} - c^m_{13})(e^r_{13} - e^m_{13})}{(K^r - K^m)}\right], \\
\varepsilon_{33} &= \varepsilon^m_{33} + f\left[\varepsilon^r_{33} - \varepsilon^m_{33} - (p-1)\frac{(e^r_{13} - e^m_{13})^2}{(K^r - K^m)}\right],
\end{aligned} \tag{10}$$

where coefficients with the superscripts r and m denote the properties of the rod and the matrix, respectively, and f is the volume fraction of the PZT-rods. The parameter p is related to the effective transverse bulk modulus K of the composite as

$$p = \frac{1}{f} \cdot \frac{K - K^m}{K^r - K^m}, \tag{11}$$

where $K^m = \left(c^m_{11} + c^m_{12}\right)/2$, $K^r = \left(c^r_{11} + c^r_{12}\right)/2$. There are other effective coefficients but they have no importance for hydrophone applications, see Avellaneda and Swart (1994) for details.

3 The optimal design problem

In this section we formulate the optimal design problems. We begin with a description of the design parameters.

3.1 *Volume fraction of the PZT rods*

All of the properties of the composite are very sensitive to the volume fraction f of the PZT rods and hence it is one of the main design variables of the problem. The previous study of Avellaneda and Swart (1994) suggested that the optimal volume fraction is small and should lie in the interval $f \in [0.05, 0.20]$. As we will see, a similar conclusion remains valid for the piezoelectric composite with a transversely isotropic matrix.

3.2 *Arrangement of the PZT rods*

As was already mentioned, given the properties and volume fractions of the rods and the matrix, the hydrophone characteristics are uniquely defined by the effective transverse bulk modulus K that depends on the spatial arrangement of the PZT rods. For any arrangement of rods, this modulus must satisfy the Hashin-Shtrikman bulk modulus bounds that for the plane problem were given by Hashin (1965),

$$K^- \le K \le K^+, \tag{12}$$

where

$$K^\pm = fK^r + (1-f)K^m - \frac{f(1-f)(K^m - K^r)^2}{fK^m + (1-f)K^r + \mu^\pm}. \tag{13}$$

Here $\mu^- = \mu^m = \left(c^m_{11} - c^m_{12}\right)/2$ and $\mu^+ = \mu^r = \left(c^r_{11} - c^r_{12}\right)/2$, ($\mu^m \le \mu^r$) are the transverse shear moduli of the matrix and PZT rods, respectively. Therefore, the microstructure of the composite can be uniquely defined (for hydrophone applications only !) by the dimensionless parameter δ as follows:

$$K = (1-\delta)\,K^{-} + \delta K^{+} . \tag{14}$$

One can treat δ as an independent design variable that completely determines the influence of the shape and distribution of the rod on the hydrophone performance characteristics.

Note also that (13) is the only place in our analyses that involves the shear moduli μ^m and μ^r .

3.3 Properties of the matrix

The hydrophone characteristics are very sensitive to the properties of the matrix material. Avellaneda and Swart (1994) showed that introducing additional porosity into the isotropic polymer matrix may dramatically improve the performance of the composite. They also showed that decreasing the Poisson's ratio of the matrix may result in enhanced performance of the composite.

Here we further explore the idea of optimally designing the matrix for the piezoelectric composite. We depart from the assumption of Avellaneda and Swart (1994) of isotropy of the matrix material and use a *transversely isotropic* matrix which itself is treated as a composite. Thus, the optimal hydrophone composite design is a two-step process: first, we create an optimal matrix by weakening the polymer by an optimal arrangement of pores. Then we embed the PZT rods into this matrix. Here we discuss the first step of this process.

We assume that the matrix material is comprised of an isotropic polymer phase with a given stiffness tensor $\mathbf{c}_p$ and a void phase (with zero stiffness). Hence, it can be viewed as an isotropic polymer which is weakened by an optimal arrangement of pores. Milton and Cherkaev (1995 a, b) raised the question as to whether the trivial bounds on the effective stiffness tensor $\mathbf{c}_m$ of a composite of voids and a phase with the properties $\mathbf{c}_p$, namely,

$$0 \le \mathbf{c}_m \le \mathbf{c}_p , \tag{15}$$

are in fact realizable. If these trivial bounds are realizable, then it follows that there exist optimal arrangements of the pores that lead to any of the transversely isotropic tensors $\mathbf{c}_m$ satisfying inequalities (15). Let us assume that this is the case. By the tensor inequality of the type $\mathbf{c}_p \ge \mathbf{c}_m$, we shall mean that the difference $\mathbf{c}_p - \mathbf{c}_m$ is a positive semidefinite matrix.

We assume that $\mathbf{c}_m$ is transversely isotropic. Although there is no theoretical lower bound for the stiffness of the matrix material, it is unrealistic to expect zero stiffness of the hydrophones to be convenient for applications. Hence we place restrictions on the matrix $\mathbf{c}_m$ via the inequalities

$$a\mu^p \le \mu^m \le \mu^p , \tag{16}$$

and

$$a\mu^m \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \le \begin{pmatrix} K^m & c^m_{13} \\ c^m_{13} & c^m_{33} \end{pmatrix} \le \begin{pmatrix} K^p & c^p_{13} \\ c^p_{13} & c^p_{33} \end{pmatrix} . \tag{17}$$

Here $\mu^m = \left(c^m_{11} - c^m_{12}\right)/2$ and $\mu^p = \left(c^p_{11} - c^p_{12}\right)/2$ are the transverse shear moduli of the matrix and the polymer, respectively, and $a < 1$ is given. Note that *the performance characteristics are extremely sensitive to the value of the parameter a.* In our numerical experiments we assume that $a = 0.03$. Decreasing a may lead to an even more dramatic increase of the values $d_h, d_h g_h$ and k_h.

We will assume that the dielectric constant $\left(\varepsilon^m_{33}\right)^T$ of the matrix material is equal to the dielectric constant $\left(\varepsilon^p_{33}\right)^T$ of the polymer. This is a reasonable assumption because we found numerically that even a variation of ε^m_{33} between the value for the polymer and that for the void space leads to only 6% variation in the values of $d_h g_h$ and k_h (d_h does not depend on the dielectric properties).

Instead of using the matrix moduli $\mathbf{K}^m, \boldsymbol{\mu}^m, \mathbf{c}^m_{13}$, we found that it is more convenient to use an alternative but equivalent set of controls, namely, the eigenvalues λ_1, λ_2 of the matrix

$$\begin{pmatrix} \mathbf{K}^m & \mathbf{c}^m_{13} \\ \mathbf{c}^m_{13} & \mathbf{c}^m_{33} \end{pmatrix} , \tag{18}$$

and the scalar parameter $x \in (-\infty, \infty)$ that describes the direction of the corresponding eigenvectors $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$ of this matrix as follows:

$$\boldsymbol{v}_1 = (-1/x \quad 1), \quad \boldsymbol{v}_2 = (x \quad 1). \tag{19}$$

The coefficients K^m, c^m_{13}, and c^m_{33} are equal to

$$K^m = \frac{\lambda_1 + x^2\lambda_2}{1+x^2}, \; c^m_{13} = -\frac{(\lambda_1 - \lambda_2)\,x}{1+x^2}, \; c^m_{33} = \frac{\lambda_1 x^2 + \lambda_2}{1+x^2}, \tag{20}$$

in terms of these new controls. One can check also that

$$E_1 = E_2 = \frac{4\left(1+x^2\right)\lambda_1\lambda_2\mu^m}{\lambda_1\lambda_2\left(1+x^2\right) + \mu^m\left(\lambda_1 x^2 + \lambda_2\right)},$$

$$E_3 = \frac{\lambda_1\lambda_2\left(1+x^2\right)}{\left(\lambda_1 + \lambda_2 x^2\right)}, \tag{21}$$

$$\nu_{12} = \nu_{21} = \frac{\lambda_1\lambda_2\left(1+x^2\right) - \mu^m\left(\lambda_1 x^2 + \lambda_2\right)}{\lambda_1\lambda_2\left(1+x^2\right) + \mu^m\left(\lambda_1 x^2 + \lambda_2\right)},$$

$$\nu_{31} = \frac{(\lambda_2 - \lambda_1)\,x}{2\left(\lambda_1 + \lambda_2 x^2\right)}, \quad \nu_{13} = \nu_{31}\frac{E_1}{E_3}, \tag{22}$$

where E_i and ν_{ij} are the corresponding Young's moduli and Poisson's ratios. These expressions will help us to interpret the results of the optimization.

In summary, the design parameters of the matrix include the three parameters λ_1, λ_2, and x [influencing the hydrophone performance directly through (10) and (11)] and a fourth parameter $\mu^m = \left(c^m_{11} - c^m_{12}\right)/2$ which enters the problem through (13)-(14). These four parameters are subjected to the restrictions (16)-(17).

3.4 Properties of the piezoceramic and the polymer

The moduli of the piezoceramic rods (indicated by the superscript r) and the polymer (indicated by the superscript p) are taken to be given by:

$$s^r_{11} = s^r_{22} = 16.4, \;\; s^r_{12} = -5.74, \;\; s^r_{13} = -7.22, \;\; s^r_{33} = 18.8,$$

$$d^r_{13} = d^r_{23} = -171, \;\; d^r_{33} = 374, \;\; \left(\varepsilon^T_{33}\right)^r = 1700\varepsilon_0, \tag{23}$$

$$s^p_{11} = s^p_{22} = s^p_{33} = 400, \;\; s^p_{12} = s^p_{13} = s^p_{23} = -148,$$

$$d^p_{13} = d^p_{23} = d^p_{33} = 0, \;\; \left(\varepsilon^T_{33}\right)^p = 3.5\varepsilon_0, \tag{24}$$

where s_{ij}^r, d_{ij}^r, and ε_{ij}^r are dyadic coefficients of the tensors $(\mathbf{s}^E)^r$, $(\mathbf{d})^r$, and $(\boldsymbol{\varepsilon}^T)^r$ that describe the PZT piezoceramic properties, and corresponding coefficients with the superscript p describing the properties of a polyurethane polymer. Here the s-coefficients are measured in the units $10^{-12}\ m^2/\mathrm{N} = \mu m^2/\mathrm{N}$, e-coefficients are measured in the units $10^{-12}\ \mathrm{C/N} = p\,\mathrm{C/N}$, and

$$\varepsilon_0 = \frac{1}{4\pi}\frac{10^{-9}}{8.98755}\frac{C^2}{\mathrm{Nm}^2} \qquad (25)$$

is the dielectric constant of the vacuum.

We are now in a position to formulate the optimal design problem. Given the parameters of the PZT ceramic (23) and the polymer (24), find the set of values of the parameters f, δ, μ^m, λ_1, λ_2, and x that maximize each of the following functionals:

(i) absolute value of the hydrostatic piezoelectric coefficient $|d_h|$;

(ii) hydrophone figure of merit $d_h g_h$;

(iii) hydrostatic electro-mechanical coupling factor k_h;

(iv) combination of the parameters $|d_h|, d_h g_h$, and k_h.

We will not precisely specify the last functional but will simply try to find the set of the design variables that keep the values of $d_h, d_h g_h$, and k_h within 15-20% of their maximal possible values.

We use the program Maple V (1981) to solve the optimization problem.

4 Numerical results and discussion

Our numerical experiments (using Maple V 1981) show that the absolute value of $|d_h|$, and also $d_h g_h$ and k_h are decreasing functions of the parameters δ, μ^m and λ_1. Therefore, these parameters were chosen to lie on their lower bounds defined by the condition $\delta \geq 0$ and by the restrictions (16) and (17), i.e.

$$\delta = 0, \quad \mu^m = a\mu^p, \quad \lambda_1 = a\mu^p. \qquad (26)$$

It is an important result, because it defines the optimal shape and arrangement of the piezoelectric rods. Namely, $\delta = 0$ means that the optimal structures are Hashin (1965) coated-cylinders assemblages with the bulk modulus K^-, see (13). Different microstructures with the same bulk modulus, consisting of a square array of the rods of special shape were discovered by Vigdergauz (1989, 1994). Recently Vigdergauz (1996) generalized his result and found the shape of the rods such that a composite containing a *hexagonal* array of these rods will be *transversely isotropic* with the transverse bulk modulus K^-. For the low volume fraction the shapes of optimal rods are very close to the circular cylinders. Therefore, *a transversely isotropic composite consisting of a hexagonal array of circular rods in a transversely isotropic polymer matrix is optimal for hydrophone designs.*

We also found that $|d_h|$, $d_h g_h$ and k_h are increasing functions of λ_2. Therefore, the parameters λ_2, and x should be chosen so as to satisfy the upper restrictions in (17) as an equality, i.e.

$$\det\begin{pmatrix} K^p - K^m & c_{13}^p - c_{13}^m \\ c_{13}^p - c_{13}^m & c_{33}^p - c_{33}^m \end{pmatrix} = 0, \qquad (27)$$

Here K^m, c_{13}^m and c_{33}^m should be expressed in terms of $\lambda_1 = a\mu^p$, λ_2, and x [cf. (20)]. Equation (27) has the solution $\lambda_2 =$

$$\mu^p\frac{x^2(3K^p - aK^p - \mu^p) - 2ax(K^p - \mu^p) + K^p(3-a) - \mu^p(1+a)}{x^2\left[K^p + \mu^p(1-a)\right] - 2x(K - \mu^p) + K^p - a\mu^p}. \qquad (28)$$

In summary, we have specified all of the design variables except the volume fraction f of the piezoceramic rods and the parameter x.

One can check that each of the functionals in the optimization problems (i)-(iii) has two maxima as functions of the parameters x at fixed f. One of these maxima corresponds to positive values of x, and the other to negative values of this parameter. By using Maple V we found all of these maxima. The results are summarized in Table 1 where we compare the performance characteristics of pure PZT ceramic to three different groups of optimal design projects.

The first row in Table 1 corresponds to the values of the parameter for the pure PZT ceramic with the moduli given by (23). Group 2 (rows 2.1-2.4 of Table 1) corresponds to a "basic" optimal design of the hydrophone made of PZT and isotropic polymer without pores. The parameters to optimize are only the shape, cross-section, and volume fraction of PZT rods *but not the elastic properties of the polymer matrix.* These basic optimal design projects correspond closely to the ones studied by Avellaneda and Swart (1994). Row 2.1 of Table 1 describes the design that gives the maximal $|d_h|$, row 2.2 describes the project that optimize $d_h g_h$, and row 2.3 optimizes k_h. Row 2.4 corresponds to a design that is good "on average", i.e. that have parameters $|d_h|, d_h g_h$, and k_h that are within 16% from their maxima in the rows 2.1-2.3.

Group 3 (rows 3.1-3.4 of Table 1) corresponds to optimal projects where we restrict ourselves to matrix materials with a positive Poisson's ratio $\nu_{31} \geq 0$ (i.e. $x \geq 0$). Good "on average" design (row 3.4) have parameters $|d_h|, d_h g_h$, and k_h equal to 73% of their maxima in the rows 3.1-3.3.

Group 4 (rows 4.1-4.4 of Table 1) corresponds to optimal projects where we do not assume that matrix Poisson's ratio ν_{31} is positive. In fact, ν_{31} is negative for all these examples. Good "on average" design (row 4.4) have parameters $|d_h|, d_h g_h$, and k_h equal to 78% of their maxima in the rows 4.1-4.3. These are more complicated structures but allow one to achieve higher hydrophone performance characteristics.

5 Optimal composites

In this section we evaluate the effective properties of optimal composites. We start with the projects 3.4 and 4.4 in Table 1. Project 3.4 is an optimal design made of a matrix with *positive* Poisson's ratio ν_{31}^m. The parameter $x = 0.21$ defines the stiffness matrix $\mathbf{c}_m$ with coefficients

$$c_{11}^m = 2.007\cdot 10^8, \quad c_{12}^m = 1.460\cdot 10^8,$$
$$c_{13}^m = 6.951\cdot 10^8, \quad c_{33}^m = 33.38\cdot 10^8, \qquad (29)$$

measured in $\mathrm{N/m}^2$. The upper three by three block of the stiffness tensor of such a material has the following eigenvalues and eigenvectors:

Table 1. Hydrophone performance characteristics of pure PZT, piezocomposite with isotropic polymer matrix, and piezocomposites with optimal transversely isotropic matrices with either positive or negative Poisson's ratios ν_{31}. An asterisk denotes the cost function(s) for the specific project

N	Project	f	x	ν_{31}^m	d_h (pC/N)	$d_h g_h$ (μm^2/N)	k_h	k_h^2
1	PZT5A:	1.000	N/A	N/A	32.0	0.068	0.078	0.0061
2	basic:							
2.1	best d_h	0.212	N/A	0.370	66.8*	1.496	0.082	0.0067
2.2	best $d_h g_h$	0.036	N/A	0.370	40.4	3.865*	0.116	0.0135
2.3	best k_h	0.041	N/A	0.370	43.3	3.848	0.116*	0.0135
2.4	good for all	0.076	N/A	0.370	56.2*	3.262*	0.110*	0.0121
3	$\nu_{31}^m \geq 0$:							
3.1	best d_h	0.109	0.124	2.513	-1087*	766	0.238	0.0566
3.2	best $d_h g_h$	0.014	0.170	2.259	-645	2200*	0.367	0.1347
3.3	best k_h	0.010	∞	0.000	352	700	0.556*	0.3091
3.4	good for all	0.028	0.210	2.005	-798*	1609*	0.407*	0.1656
4	$\nu_{31}^m \leq 0$:							
4.1	best d_h	0.098	-0.138	-2.107	1458*	1517	0.302	0.0912
4.2	best $d_h g_h$	0.006	-0.350	-1.240	762	5655*	0.458	0.2098
4.3	best k_h	0.017	-1.800	-0.270	537	1508	0.567*	0.3215
4.4	good for all	0.021	-0.200	-1.817	1147*	4445*	0.448*	0.2007

$$
\begin{aligned}
\rho_1 &= 36.32 \cdot 10^8 , & \boldsymbol{v}^{(1)} &= (\ 0.203 \ \ 0.203 \ \ 0.958\ ), \\
\rho_2 &= 0.547 \cdot 10^8 , & \boldsymbol{v}^{(2)} &= (\ -0.707 \ \ 0.707 \ \ 0.0\ ), \\
\rho_3 &= 0.525 \cdot 10^8 , & \boldsymbol{v}^{(3)} &= (\ 0.677 \ \ 0.677 \ \ -0.286\ ).
\end{aligned} \tag{30}
$$

It is seen that one eigenvalue ($\rho_1 = 36.32 \cdot 10^8$) is of the order of the stiffness of the polymer (c_{11}^p=44.22$\cdot 10^8$) but the other two eigenvalues are significantly lower. This material is transversely isotropic; it can be easily deformed by shear $\boldsymbol{v}_2$ in the $x_1 - x_2$ plane, and in $\boldsymbol{v}^{(3)}$ direction, but strongly resists deformation in the $\boldsymbol{v}^{(1)}$ direction.

For project 4.4, $x = -0.20$ and the components of the stiffness matrix $\mathbf{c}_m$ are given by

$$
\begin{aligned}
c_{11}^m &= \ 1.276 \cdot 10^8 , \quad c_{12}^m = 0.729 \cdot 10^8 , \\
c_{13}^m &= -3.644 \cdot 10^8 , \quad c_{33}^m = 18.49 \cdot 10^8 .
\end{aligned} \tag{31}
$$

The upper three by three block of this stiffness tensor has the following eigenvalues and eigenvectors:

$$
\begin{aligned}
\rho_1 &= 19.97 \cdot 10^8 , & \boldsymbol{v}^{(1)} &= (\ 0.195 \ \ 0.195 \ \ -0.961\ ), \\
\rho_2 &= 0.547 \cdot 10^8 , & \boldsymbol{v}^{(2)} &= (\ -0.707 \ \ 0.707 \ \ 0.0\ ), \\
\rho_3 &= 0.527 \cdot 10^8 , & \boldsymbol{v}^{(3)} &= (\ 0.680 \ \ 0.680 \ \ 0.276\ ).
\end{aligned} \tag{32}
$$

Again, one of these eigenvalues ($\rho_1 = 19.97 \cdot 10^8$) is of the order of the stiffness of the polymer ($c_{11}^p = 44.22 \cdot 10^8$), and the other two are significantly lower. This material is also transversely isotropic; it can be easily deformed by shear in the $x_1 - x_2$ plane, and in the $\boldsymbol{v}^{(3)}$ direction, but strongly resists deformation in the $\boldsymbol{v}^{(1)}$ direction.

Let us now turn our attention to project 3.3. This has an interesting feature in that the optimal value of the parameter x is equal to infinity, and the optimal value of the Poisson's ratios $\nu_{13}^m = \nu_{31}^m = 0$ is equal to zero. The stiffness matrix $\mathbf{c}_m$ for this project has the coefficients

$$
\begin{aligned}
c_{11}^m &= 20.02 \cdot 10^8 , \ c_{12}^m = 19.47 \cdot 10^8 , \\
c_{13}^m &= \ 0.0 , \ c_{33}^m = 0.2737 \cdot 10^8 ,
\end{aligned} \tag{33}
$$

and the following eigenvalues and eigenvectors of the upper three by three block of the stiffness matrix:

$$
\begin{aligned}
\rho_1 &= 39.49 \cdot 10^8 , & \boldsymbol{v}^{(1)} &= (\ 0.707 \ \ 0.707 \ \ 0\ ), \\
\rho_2 &= 0.547 \cdot 10^8 , & \boldsymbol{v}^{(2)} &= (\ -0.707 \ \ 0.707 \ \ 0\ ), \\
\rho_3 &= 0.274 \cdot 10^8 , & \boldsymbol{v}^{(3)} &= (\ 0 \ \ 0 \ \ 1\ ).
\end{aligned} \tag{34}
$$

As can be seen, such a material resists only compression in the $x_1 - x_2$ plane. It can be approximately modelled by a laminate material with a polymer and void layers alternating along the x_3-direction. In order to ensure a small stiffness in the x_3-directions, one needs to connect the polymer layers by small polymer rods. Numerical experiments show that the performance of the piezocomposite with a matrix made of such a laminate composite almost matches the performance of the piezocomposite with the optimal matrix (33).

6 Summary of the results

Our investigation allows us to formulate the following results.

- Optimal piezocomposites consist of *hexagonal arrays* of piezoelectric rods in a transversely isotropic polymer matrix.
- The effective tensors of the matrix are assumed to be transversely isotropic. The optimal matrix material is *highly anisotropic*, with one eigenvalue of the stiffness matrix being as large as possible, and the others being as small as allowed by the design restrictions. Moreover, matrix materials with *negative enspace Poisson's ratios* $\nu_{31} \leq 0$ deliver better results.
- The volume fraction of the piezoelectric rods should be small; it varies in the interval $f \in [0.01, 0.11]$ depending on the particular functional (see Table 1).
- The optimal design is very sensitive to the variation of the volume fraction of the rods and properties of the polymer matrix.
- Optimal designs greatly enhance all of the performance characteristic of the hydrophone (see Table 1).
- Our results give a theoretical upper bound on the hydrophone characteristics, i.e. the actual performance will be lower than our estimate. Nevertheless, we consider

these bounds to be very helpful, because they allow one to estimate the possible enhancement of the hydrophone performance due to structural optimization.

- Although we have not specified the microstructures, our results provide a helpful guideline as to the basic features of the effective properties of optimal piezocomposites, such as extreme anisotropy and optimal direction of the "stiff" mode of the elasticity tensor of the optimal matrix material.

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